1-1 Enrichment

Toothpick Triangles

Variable expressions can be used to represent patterns and help solve problems. Consider the problem of creating triangles out of toothpicks shown below.

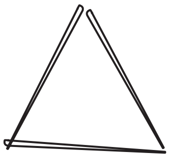

Figure 1

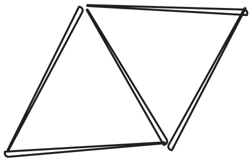

Figure 2

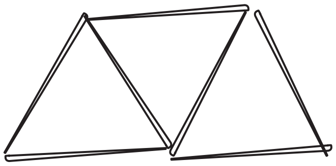

Figure 3

1. How many toothpicks does it take to create each figure?

2. How many toothpicks does it take to make up the perimeter of each image?

3. Sketch the next three figures in the pattern.

4. Continue the pattern to complete the table.

Image Number	1	2	3	4	5	6	7	8	9	10
Number of toothpicks	3	5	7							
Number of toothpicks in Perimeter	3	4	5							

5. Let the variable n represent the figure number. Write an expression that can be used to find the number of toothpicks needed to create figure n.

6. Let the variable n represent the figure number. Write an expression that can be used to find the number of toothpicks in the perimeter of figure n.

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1-2 Enrichment

The Four Digits Problem

One well-known mathematic problem is to write expressions for consecutive numbers beginning with 1. On this page, you will use the digits 1, 2, 3, and 4. Each digit is used only once. You may use addition, subtraction, multiplication (not division), exponents, and parentheses in any way you wish. Also, you can use two digits to make one number, such as 12 or 34.

Express each number as a combination of the digits 1, 2, 3, and 4.

$1 = (3 \times 1) - (4 - 2)$
2 = ____________________
3 = ____________________
4 = ____________________
5 = ____________________
6 = ____________________
7 = ____________________
8 = ____________________
9 = ____________________
10 = ____________________
11 = ____________________
12 = ____________________
13 = ____________________
14 = ____________________
15 = ____________________
16 = ____________________
17 = ____________________

18 = ____________________
$19 = 3(2 + 4) + 1$
20 = ____________________
21 = ____________________
22 = ____________________
$23 = 31 - (4 \times 2)$
24 = ____________________
25 = ____________________
26 = ____________________
27 = ____________________
28 = ____________________
29 = ____________________
30 = ____________________
31 = ____________________
32 = ____________________
33 = ____________________
34 = ____________________

$35 = 2^{(4+1)} + 3$
36 = ____________________
37 = ____________________
38 = ____________________
39 = ____________________
40 = ____________________
41 = ____________________
42 = ____________________
$43 = 42 + 1^3$
44 = ____________________
45 = ____________________
46 = ____________________
47 = ____________________
48 = ____________________
49 = ____________________
50 = ____________________

Does a calculator help in solving these types of puzzles? Give reasons for your opinion.

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____

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1-3 Enrichment

Solution Sets

Consider the following open sentence.

It is the name of a month between March and July.

You know that a replacement for the variable *It* must be found in order to determine if the sentence is true or false. If *It* is replaced by either April, May, or June, the sentence is true.
The set {April, May, June} is called the solution set of the open sentence given above. This set includes all replacements for the variable that make the sentence true.

Write the solution set for each open sentence.

1. It is the name of a state beginning with the letter A.

2. It is a primary color.

3. Its capital is Harrisburg.

4. It is a New England state.

5. $x + 4 = 10$

6. It is the name of a month that contains the letter *r*.

7. During the 1990s, she was the wife of a U.S. President.

8. It is an even number between 1 and 13.

9. $31 = 72 - k$

10. It is the square of 2, 3, or 4.

Write an open sentence for each solution set.

11. {A, E, I, O, U}

12. {1, 3, 5, 7, 9}

13. {June, July, August}

14. {Atlantic, Pacific, Indian, Arctic}

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1-4 Enrichment

Closure

A *binary operation* matches two numbers in a set to just one number. Addition is a binary operation on the set of whole numbers. It matches two numbers such as 4 and 5 to a single number, their sum.

If the result of a binary operation is always a member of the original set, the set is said to be *closed* under the operation. For example, the set of whole numbers is closed under addition because $4 + 5$ is a whole number. The set of whole numbers is not closed under subtraction because $4 - 5$ is not a whole number.

Tell whether each operation is binary. Write *yes* or *no*.

1. the operation ↲, where a ↲ b means to choose the lesser number from a and b
2. the operation ©, where a © b means to cube the sum of a and b
3. the operation sq, where $sq(a)$ means to square the number a
4. the operation exp, where $exp(a, b)$ means to find the value of a^b
5. the operation $\Uparrow$, where $a \Uparrow b$ means to match a and b to any number greater than either number
6. the operation $\Rightarrow$, where $a \Rightarrow b$ means to round the product of a and b up to the nearest 10

Tell whether each set is closed under addition. Write *yes* or *no*. If your answer is no, give an example.

7. even numbers
8. odd numbers
9. multiples of 3
10. multiples of 5
11. prime numbers
12. nonprime numbers

Tell whether the set of whole numbers is closed under each operation. Write *yes* or *no*. If your answer is no, give an example.

13. multiplication: $a \times b$
14. division: $a \div b$
15. exponentiation: ab
16. squaring the sum: $(a + b)^2$

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1-5 Enrichment

The Maya

0	𝋠	10	𝋪
1	𝋡	11	𝋫
2	𝋢	12	𝋬
3	𝋣	13	𝋭
4	𝋤	14	𝋮
5	𝋥	15	𝋯
6	𝋦	16	𝋰
7	𝋧	17	𝋱
8	𝋨	18	𝋲
9	𝋩	19	𝋳

The Maya were a Native American people who lived from about 1500 B.C. to about 1500 A.D. in the region that today encompasses much of Central America and southern Mexico. Their many accomplishments include exceptional architecture, pottery, painting, and sculpture, as well as significant advances in the fields of astronomy and mathematics.

The Maya developed a system of numeration that was based on the number twenty. The basic symbols of this system are shown in the table at the right. The places in a Mayan numeral are written vertically—the bottom place represents *ones*, the place above represents *twenties*, the place above that represents 20 × 20, or *four hundreds*, and so on. For instance, this is how to write the number 997 in Mayan numerals.

$$\begin{array}{cclclr} \text{𝋢} & \leftarrow & 2 \times & \boxed{400} & = & 800 \\ \text{𝋩} & \leftarrow & 9 \times & \boxed{20} & = & 180 \\ \text{𝋱} & \leftarrow & 17 \times & \boxed{1} & = & \underline{17} \\ & & & & & 997 \end{array}$$

Evaluate each expression when *v* = 𝋦, *w* = 𝋲, *x* = 𝋤, *y* = 𝋠, and *z* = 𝋬. Then write the answer in Mayan numerals. Exercise 5 is done for you.

1. $\frac{z}{w}$

2. $\frac{v + w + z}{x}$

3. xv

4. vxy

5. $wx - z$ $\begin{matrix}\text{𝋣}\\\text{𝋠}\end{matrix}$

6. $vz + xy$

7. $w(v + x + z)$

8. vwz

9. $z(wx - x)$

Tell whether each statement is *true* or *false*.

10. 𝋭 + 𝋦 = 𝋦 + 𝋭

11. $\begin{matrix}\text{𝋭}\\\text{𝋦}\end{matrix} = \begin{matrix}\text{𝋦}\\\text{𝋭}\end{matrix}$

12. $\begin{matrix}\text{𝋨}\\\text{𝋪}\end{matrix} = \begin{matrix}\text{𝋭}\\\text{𝋥}\end{matrix}$

13. (𝋣 + 𝋥) + 𝋪 = 𝋣 + (𝋥 + 𝋪)

14. How are Exercises 10 and 11 alike? How are they different?

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1-6 Enrichment

Properties of Operations

Let's make up a new operation and denote it by $\circledast$, so that $a \circledast b$ means b^a.

$2 \circledast 3 = 3^2 = 9$

$(1 \circledast 2) \circledast 3 = 2^1 \circledast 3 = 3^2 = 9$

1. What number is represented by $2 \circledast 3$?

2. What number is represented by $3 \circledast 2$?

3. Does the operation $\circledast$ appear to be commutative?

4. What number is represented by $(2 \circledast 1) \circledast 3$?

5. What number is represented by $2 \circledast (1 \circledast 3)$?

6. Does the operation $\circledast$ appear to be associative?

Let's make up another operation and denote it by $\oplus$, so that $a \oplus b = (a + 1)(b + 1)$.

$3 \oplus 2 = (3 + 1)(2 + 1) = 4 \cdot 3 = 12$

$(1 \oplus 2) \oplus 3 = (2 \cdot 3) \oplus 3 = 6 \oplus 3 = 7 \cdot 4 = 28$

7. What number is represented by $2 \oplus 3$?

8. What number is represented by $3 \oplus 2$?

9. Does the operation $\oplus$ appear to be commutative?

10. What number is represented by $(2 \oplus 3) \oplus 4$?

11. What number is represented by $2 \oplus (3 \oplus 4)$?

12. Does the operation $\oplus$ appear to be associative?

13. What number is represented by $1 \circledast (3 \oplus 2)$?

14. What number is represented by $(1 \circledast 3) \oplus (1 \circledast 2)$?

15. Does the operation $\circledast$ appear to be distributive over the operation $\oplus$?

16. Let's explore these operations a little further. What number is represented by $3 \circledast (4 \oplus 2)$?

17. What number is represented by $(3 \circledast 4) \oplus (3 \circledast 2)$?

18. Is the operation $\circledast$ actually distributive over the operation $\oplus$?

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1-7 Enrichment

Counterexamples

Some statements in mathematics can be proven false by **counterexamples**. Consider the following statement.

For any numbers a and b, $a - b = b - a$.

You can prove that this statement is false in general if you can find one example for which the statement is false.

Let $a = 7$ and $b = 3$. Substitute these values in the equation above.

$$7 - 3 \stackrel{?}{=} 3 - 7$$
$$4 \neq -4$$

In general, for any numbers a and b, the statement $a - b = b - a$ is false. You can make the equivalent verbal statement: subtraction is *not* a commutative operation.

In each of the following exercises *a*, *b*, and *c* are any numbers. Prove that the statement is false by counterexample.

1. $a - (b - c) \stackrel{?}{=} (a - b) - c$

2. $a \div (b \div c) \stackrel{?}{=} (a \div b) \div c$

3. $a \div b \stackrel{?}{=} b \div a$

4. $a \div (b + c) \stackrel{?}{=} (a \div b) + (a \div c)$

5. $a + (bc) \stackrel{?}{=} (a + b)(a + c)$

6. $a^2 + a^2 \stackrel{?}{=} a^4$

7. Write the verbal equivalents for Exercises 1, 2, and 3.

8. For the distributive property $a(b + c) = ab + ac$ it is said that multiplication distributes over addition. Exercises 4 and 5 prove that some operations do not distribute. Write a statement for each exercise that indicates this.

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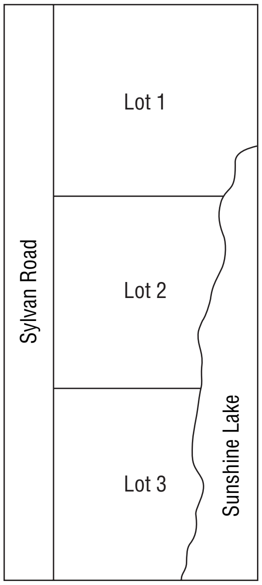
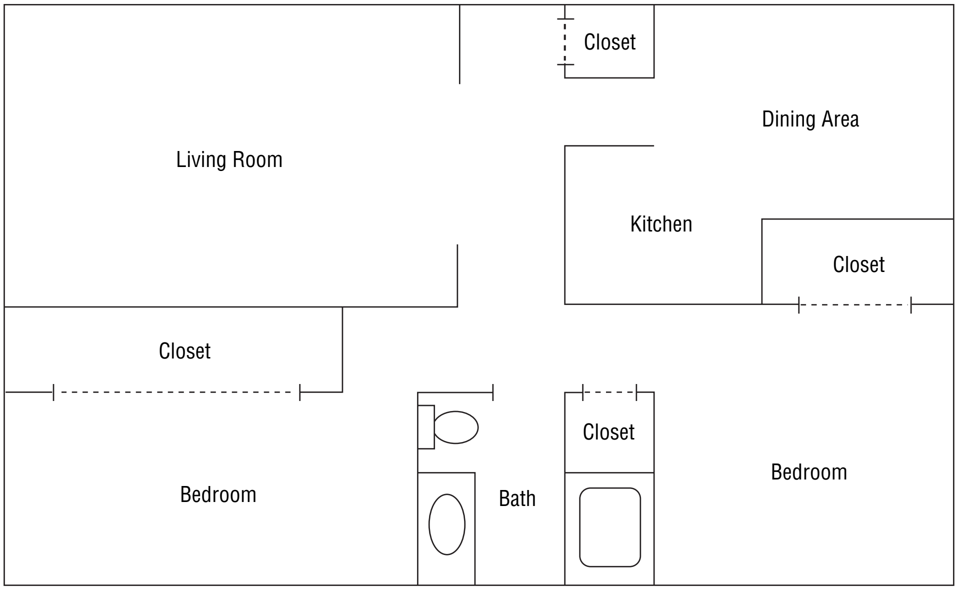
1-8 Enrichment

Scale Drawings

The map at the left below shows building lots for sale. The scale ratio is 1:2400. At the right below is the floor plan for a two-bedroom apartment. The length of the living room is 6 m. On the plan the living room is 6 cm long.





Answer each question.

1. On the map, how many feet are represented by an inch?

2. On the map, measure the frontage of Lot 2 on Sylvan Road in inches. What is the actual frontage in feet?

3. What is the scale ratio represented on the floor plan?

4. On the floor plan, measure the width of the living room in centimeters. What is the actual width in meters?

5. About how many square meters of carpeting would be needed to carpet the living room?

6. Make a scale drawing of your classroom using an appropriate scale.

7. Use your scale drawing to determine how many square meters of tile would be needed to install a new floor in your classroom.

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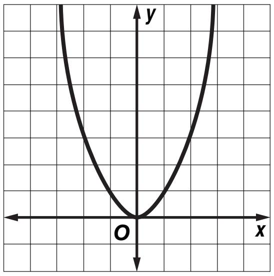
1-9 Enrichment

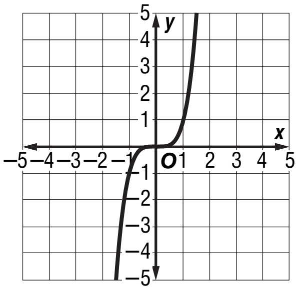
Even and Odd Functions

We know that numbers can be either even or odd. It is also true that functions can be defined as even or odd. For a function to be even means that it is symmetric about the y-axis. That is, if you fold the graph along the y-axis, the two halves of the graph match exactly. For a function to be odd means that the function is symmetric about the origin. This means if you rotate the graph using the origin as the center, it will match its original position before completing a full turn.

The function $y = x^2$ is an even function.

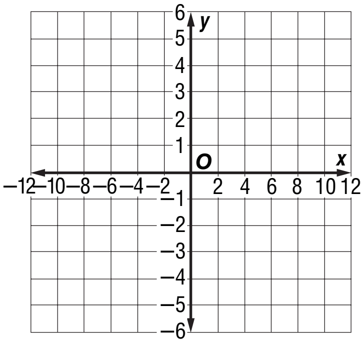


The function $y = x^5$ is an odd function. If you rotate the graph 180º the graph will lie on itself.



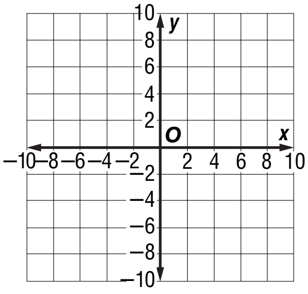
1. The table below shows the ordered pairs of an even function. Graph the points and sketch the graph. Complete the table.

x	−12	−5	−1	1	5	12
y	6	3	1			



2. The table below shows the ordered pairs of an odd function. Graph the points and sketch the graph. Complete the table.

x	−10	−4	−2	2	4	10
y	8	4	2			



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2-1 Enrichment

Guess the Number

Think of a number. Add five to your number. Now, double your result. Double your result again. Divide you answer by four. Finally, subtract your original number. Your result is five.

How is it possible to know what the answer is without knowing the original number? Write the steps listed above as an expression in equation form. Then use algebra to show why this trick works.

Think of a number: x

Add five to your number: $x + 5$

Double your result: $2(x + 5)$

Double your result again: $2(2(x + 5))$

Divide you answer by four: $\frac{2(2(x + 5))}{4}$

Subtract your original number: $\frac{2(2(x + 5))}{4} - x$

Simplify the final expression: $\frac{4(x + 5)}{4} - x$ Multiply.

$x + 5 - x$ Divide.

5 Simplify.

So, the result will always be five, no matter what the starting number is.

Write variable expressions to determine why each number trick works.

1. Think of a number. Add eight. Double your result. Next, subtract 16. Finally, divide your result by 2. You get your original number back.

2. Think of a number. Multiply by 10. Add 5 to your result. Next, subtract 3. Then add 2. Next, subtract 4. Divide your result by 5. Finally, subtract your original number. Your result is your original number.

3. Think of a number. Add 1. Multiply your result by 6. Now, double your result. Next, divide your result by 12. Finally, subtract your original number. Your result is 1.

4. Think of a number. Multiply by 5. Add five to your result. Now, divide by 5. Subtract 1 from your result. Finally, subtract your original number. Your final result is 0.

5. Think of a number. Add 30. Multiply by 3. Multiply again by 2. Divide your result by 6. Finally, subtract your original number. Your answer is 30.

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2-2 Enrichment

Elevator Puzzle

Jose gets on the elevator and rides without pushing any buttons. First, the elevator goes up 4 floors where Bob gets on. Bob goes down 6 floors and gets off. At that same floor Florence gets on and goes up one floor before getting off. The elevator then moves down 8 floors to pick up the Hartt family who ride down 3 floors and get off. Then the elevator goes up one floor, picks up Kris, and goes down 6 floors to the street level where Jose exits the elevator.

1. Suppose x is your starting point. Write an equation that represents Jose's elevator ride.

2. At what floor did Jose get on the elevator?

Now that you know the starting point of Jose, the starting point of every other person who rode the elevator can be determined.

3. At what floor did Bob get on the elevator? At what floor did Bob get off?

4. At what floor did Florence get on the elevator? At what floor did Florence get off?

5. At what floor did the Hartt family get on the elevator? At what floor did the Hartt family get off?

6. At what floor did Kris get on the elevator? At what floor did Kris get off?

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2-3 Enrichment

Using Right Triangles

Right triangles can be used to solve problems. To use right triangles to solve shadow problems, compare the height of the larger object to the height of the smaller object and set that ratio equal to the ratio formed when comparing the length of the shadow of the larger object to the length of the shadow of the smaller object.

Example **Use a triangle to determine how tall a tree is if the tree casts a 30 foot shadow at the same time of day that a yardstick casts an 8 foot shadow.**

Step 1: Set up the comparisons.

$\frac{x}{3} = \frac{30}{8}$ Compare the height of the tree to the height of the yardstick.

Step 2: Solve.

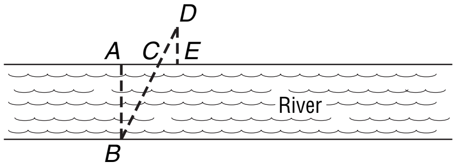
$8x = 90$ Multiply both sides by the LCD, 24.

$x = 11.25$ feet Divide both sides by 8.

Exercises

Use triangles to solve each problem.

1. You want to know the distance across the river in the figure below without crossing the river. If $AC = 10$ m; $CE = 4$ m; $ED = 10$ m, find the distance AB across the river. (Note: Point C is on line segment AE and line segment BD.).



2. A person who is 6 feet tall is looking for a flagpole that is 50 feet tall. If the person's shadow is 7 feet long, what is the length of the shadow of the 50-foot flagpole?

3. Ting and Maria are trying to find a way to guess how long each other's shadow is? Ting is $6\frac{1}{4}$ feet tall and Maria is $5\frac{1}{2}$ feet tall. If Maria stands at the shadow top of Ting's head, their two combined shadows total 18 feet. How long is each shadow? (round answer to the nearest inch.)

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2-4 Enrichment

Consecutive Integer Problems

Many types of problems and puzzles involve the idea of consecutive integers. Knowing how to represent these integers algebraically can help to solve the problem.

Example **Find four consecutive odd integers whose sum is −80.**

An odd integer can be written as $2n + 1$, where n is any integer.

If 2n + 1 is the first odd integer, then add 2 to get the next largest odd integer, and so on.

Now write an equation to solve this problem.

$(2n + 1) + (2n + 3) + (2n + 5) + (2n + 7) = -80$

Exercises

Write an equation for each problem. Then solve.

1. Complete the solution to the problem in the example.

2. Find three consecutive even integers whose sum is 132.

3. Find two consecutive integers whose sum is 19.

4. Find two consecutive integers whose sum is 100.

5. The lesser of two consecutive even integers is 10 more than one-half the greater. Find the integers.

6. The greater of two consecutive even integers is 6 less than three times the lesser. Find the integers.

7. Find four consecutive integers such that twice the sum of the two greater integers exceeds three times the first by 91.

8. Find a set of four consecutive positive integers such that the greatest integer in the set is twice the least integer in the set.

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2-5 Enrichment

Identities

An equation that is true for every value of the variable is called an **identity**. When you try to solve an identity, you end up with a statement that is always true. Here is an example.

Example **Solve $8 - (5 - 6x) = 3(1 + 2x)$.**

$$8 - (5 - 6x) = 3(1 + 2x)$$
$$8 - 5 - (-6x) = 3(1 + 2x)$$
$$8 - 5 + 6x = 3 + 6x$$
$$3 + 6x = 3 + 6x$$

Exercises

State whether each equation is an identity. If it is not, find its solution.

1. $2(2 - 3x) = 3(3 + x) + 4$

2. $5(m + 1) + 6 = 3(4 + m) + (2m - 1)$

3. $(5t + 9) - (3t - 13) = 2(11 + t)$

4. $14 - (6 - 3c) = 4c - c$

5. $3y - 2(y + 19) = 9y - 3(9 - y)$

6. $3(3h - 1) = 4(h + 3)$

7. Use the true equation $3x - 2 = 3x - 2$ to create an identity of your own.

8. Use the false equation $1 = 2$ to create an equation with no solution.

9. Create an equation whose solution is $x = 3$.

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2-6 Enrichment

Angles of a Triangle

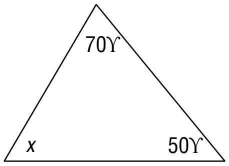
In geometry, many statements about physical space are proven to be true. Such statements are called **theorems**. Here are two examples of geometric theorems.

a. The sum of the measures of the angles of a triangle is 180°.

b. If two sides of a triangle have equal measure, then the two angles opposite those sides also have equal measure.

For each of the triangles, write an equation and then solve for *x*. (A tick mark on two or more sides of a triangle indicates that the sides have equal measure.)

1.



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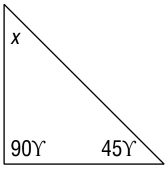

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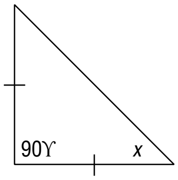

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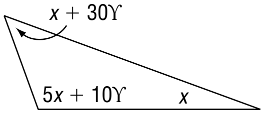

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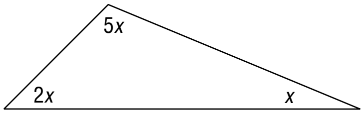

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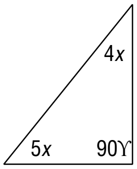

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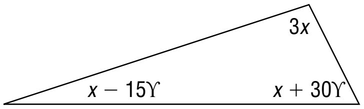

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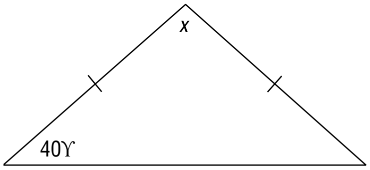

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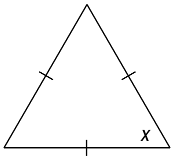
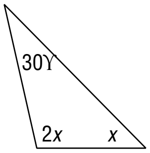


10.



11. Two angles of a triangle have the same measure. The sum of the measures of these angles is one-half the measure of the third angle. Find the measures of the angles of the triangle.

12. The measure of one angle of a triangle is twice the measure of a second angle. The measure of the third angle is 12 less than the sum of the other two. Find the measures of the angles of the triangle.

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2-7 Enrichment

Using Percent

Use what you have learned about percent to solve each problem.

A TV movie had a “rating” of 15 and a 25 “share.” The rating is the percentage of the nation’s total TV households that were tuned in to this show. The share is the percentage of homes with TVs turned on that were tuned to the movie. How many TV households had their TVs turned off at this time?

To find out, let T = the number of TV households
and x = the number of TV households with the TV off.
Then $T - x$ = the number of TV households with the TV on.

Since $0.15T$ and $0.25(T - x)$ both represent the number of households tuned to the movie,

$$0.15T = 0.25(T - x)$$
$$0.15T = 0.25T - 0.25x.$$

Solve for x. $0.25x = 0.10T$

$$x = \frac{0.10T}{0.25} = 0.40T$$

Forty percent of the TV households had their TVs off when the movie was aired.

Answer each question.

1. During that same week, a sports broadcast had a rating of 22.1 and a 43 share. Show that the percent of TV households with their TVs off was about 48.6%.

2. Find the percent of TV households with their TVs turned off during a show with a rating of 18.9 and a 29 share.

3. Show that if T is the number of TV households, r is the rating, and s is the share, then the number of TV households with the TV off is $\frac{(s - r)T}{s}$.

4. If the fraction of TV households with no TV on is $\frac{s - r}{s}$ then show that the fraction of TV households with TVs on is $\frac{r}{s}$.

5. Find the percent of TV households with TVs on during the most watched serial program in history: the last episode of M*A*S*H, which had a 60.3 rating and a 77 share.

6. A local station now has a 2 share. Each share is worth $50,000 in advertising revenue per month. The station is thinking of going commercial free for the three months of summer to gain more listeners. What would its new share have to be for the last 4 months of the year to make more money for the year than it would have made had it not gone commercial free?

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2-8 Enrichment

Compound Interest

In most banks, interest on savings accounts is compounded at set time periods such as three or six months. At the end of each period, the bank adds the interest earned to the account. During the next period, the bank pays interest on all the money in the bank, including interest. In this way, the account earns interest on interest.

Suppose Ms. Tanner has $1000 in an account that is compounded quarterly at 5%. Find the balance after the first two quarters.

Use $I = prt$ to find the interest earned in the first quarter if $p = 1000$ and $r = 5\%$. Why is t equal to $\frac{1}{4}$?

First quarter: $I = 1000 \times 0.05 \times \frac{1}{4}$

$I = 12.50$

The interest, $12.50, earned in the first quarter is added to $1000. The principal becomes $1012.50.

Second quarter: $I = 1012.50 \times 0.05 \times \frac{1}{4}$

$I = 12.65625$

The interest in the second quarter is $12.66.

The balance after two quarters is $1012.50 + 12.66 or $1025.16.

Answer each of the following questions.

1. How much interest is earned in the third quarter of Ms. Tanner's account?

2. What is the balance in her account after three quarters?

3. How much interest is earned at the end of one year?

4. What is the balance in her account after one year?

5. Suppose Ms. Tanner's account is compounded semiannually. What is the balance at the end of six months?

6. What is the balance after one year if her account is compounded semiannually?

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2-9 Enrichment

Expected Value

Expected value is the average return of an event over repeated trial. Expected value is also a form of weighted average and can be used to determine whether or not you should play a game based on what the expected value of your winnings is. It can also be used to determine the expected length of a game.

Example **You are rolling a number cube to determine the number of candy bars you will win at a school fair. If you roll a one, you get 12 candy bars. If you roll a two or a three, you get 3 candy bars. If you roll a four, five or six, you get 2 candy bars.**

Since there are six possibilities for what you can roll:

- $\frac{1}{6}$ of the time, you will win 12 candy bars.
- $\frac{2}{6}$ or $\frac{1}{3}$ of the time, you will win 3 candy bars.
- $\frac{3}{6}$ or $\frac{1}{2}$ of the time, you will win 2 candy bars.

The expected value is $\frac{1}{6}(12) + \frac{1}{3}(3) + \frac{1}{2}(2)$ or 4 candy bars.

Therefore, if you played the game 100 times, you could expect to win about 4 candy bars each time.

Exercises

Find the expected value of each of the following events.

1. You are flipping a coin. If you flip heads, you win \$5.00. If you flip tails, you win \$1.00.

2. You are rolling a number cube to determine the number of candy bars you will win at a school fair. If you roll a 1, you get 18 candy bars. If you roll a 2 or a 3, you get 6 candy bars. If you roll a 4, 5 or 6, you get 0 candy bars.

3. You are rolling a number cube to determine the amount of money you will win at a school fair. If you roll a 1, you get \$12.00. If you roll a 2 or a 3, you get \$3.00. If you roll a 4, 5 or 6, you owe \$2.00.

4. You are flipping a coin. If you flip heads, you win \$10. If you flip tails, you win \$2.

5. You are rolling a number cube to determine the number of candy bars you will win at a school fair. If you roll a 1, you get 24 candy bars. If you roll a 2 or a 3, you get 6 candy bars. If you roll a 4, 5 or 6, you get none.

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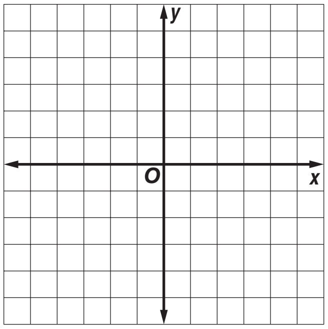
3-1 Enrichment

Inverse Relations

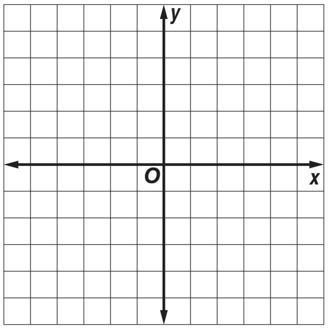
On each grid below, plot the points in Sets A and B. Then connect the points in Set A with the corresponding points in Set B. Then find the inverses of Set A and Set B, plot the two sets, and connect those points.



Set A	Set B
(−4, 0)	(0, 1)
(−3, 0)	(0, 2)
(−2, 0)	(0, 3)
(−1, 0)	(0, 4)

Inverse

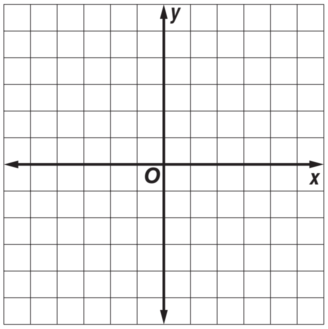
	Set A	Set B
1.		
2.		
3.		
4.		



Set A	Set B
(−3, −3)	(−2, 1)
(−2, −2)	(−1, 2)
(−1, −1)	(0, 3)
(0, 0)	(1, 4)

Inverse

	Set A	Set B
5.		
6.		
7.		
8.		



Set A	Set B
(−4, 1)	(3, 2)
(−3, 2)	(3, 2)
(−2, 3)	(3, 2)
(−1, 4)	(3, 2)

Inverse

	Set A	Set B
9.		
10.		
11.		
12.		

13. What is the graphical relationship between the line segments you drew connecting points in Sets A and B and the line segments connecting points in the inverses of those two sets?

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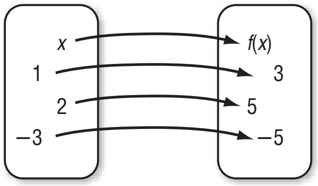
3-2 Enrichment

Composite Functions

Three things are needed to have a function—a set called the domain, a set called the range, and a rule that matches each element in the domain with only one element in the range. Here is an example.

Rule: $f(x) = 2x + 1$



$f(x) = 2x + 1$

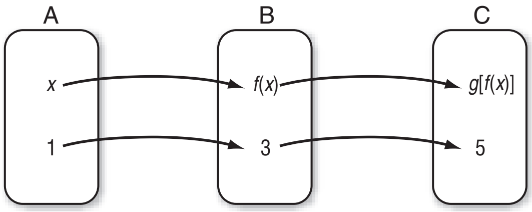
$f(1) = 2(1) + 1 = 2 + 1 = 3$

$f(2) = 2(2) + 1 = 4 + 1 = 5$

$f(-3) = 2(-3) + 1 = -6 + 1 = -5$

Suppose we have three sets A, B, and C and two functions described as shown below.

Rule: $f(x) = 2x + 1$ Rule: $g(y) = 3y - 4$



$g(y) = 3y - 4$

$g(3) = 3(3) - 4 = 5$

Let's find a rule that will match elements of set A with elements of set C without finding any elements in set B. In other words, let's find a rule for the **composite function $g[f(x)]$**.

Since $f(x) = 2x + 1$, $g[f(x)] = g(2x + 1)$.

Since $g(y) = 3y - 4$, $g(2x + 1) = 3(2x + 1) - 4$, or $6x - 1$.

Therefore, $g[f(x)] = 6x - 1$.

Find a rule for the composite function $g[f(x)]$.

1. $f(x) = 3x$ and $g(y) = 2y + 1$

2. $f(x) = x^2 + 1$ and $g(y) = 4y$

3. $f(x) = -2x$ and $g(y) = y^2 - 3y$

4. $f(x) = \dfrac{1}{x - 3}$ and $g(y) = y^{-1}$

5. Is it always the case that $g[f(x)] = f[g(x)]$? Justify your answer.

3-3 Enrichment

Translating Linear Graphs

Linear graphs can be **translated** on the coordinate plane. This means that the graph moves up, down, right, or left without changing its direction.

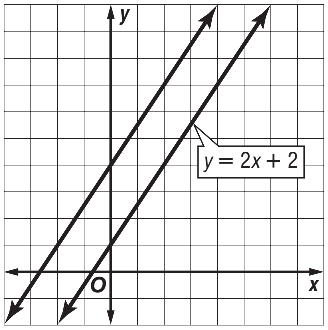
Translating the graphs up or down affects the *y*-coordinate for a given *x* value. Translating the graph right or left affects the *x*-coordinate for a given *y* value.

Example **Translate the graph of $y = 2x + 3$, 3 units up.**

$y = 2x + 2$	
x	y
−1	0
0	2
1	4
2	6

Add 3 to each y value.

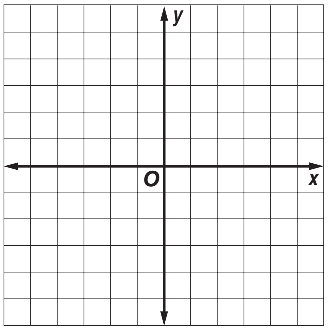
Translation	
x	y
−1	3
0	5
1	7
2	9



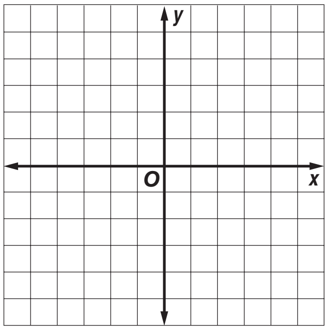
Exercises

Graph the function and the translation on the same coordinate plane.

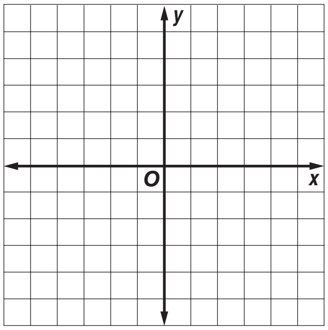
1. $y = x + 4$, 3 units down



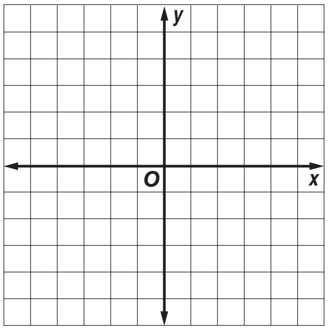
2. $y = 2x - 2$, 2 units left



3. $y = -2x + 1$, 1 unit right



4. $y = -x - 3$, 2 units up



3-4 Enrichment

Arithmetic Series

An arithmetic series is a series in which each term after the first may be found by adding the same number to the preceding term. Let S stand for the following series in which each term is 3 more than the preceding one.

$S = 2 + 5 + 8 + 11 + 14 + 17 + 20$

The series remains the same if we reverse the order of all the terms. So let us reverse the order of the terms and add one series to the other, term by term. This is shown at the right.

$$\begin{aligned} S &= 2 + 5 + 8 + 11 + 14 + 17 + 20 \\ S &= 20 + 17 + 14 + 11 + 8 + 5 + 2 \\ 2S &= 22 + 22 + 22 + 22 + 22 + 22 + 22 \\ 2S &= 7(22) \\ S &= \frac{7(22)}{2} = 7(11) = 77 \end{aligned}$$

Let a represent the first term of the series.

Let ℓ represent the last term of the series.

Let n represent the number of terms in the series.

In the preceding example, a = 2, l = 20, and $n = 7$. Notice that when you add the two series, term by term, the sum of each pair of terms is 22. That sum can be found by adding the first and last terms, 2 + 20 or $a + \ell$. Notice also that there are 7, or n, such sums. Therefore, the value of $2S$ is 7(22), or $n(a + \ell)$ in the general case. Since this is twice the sum of the series, you can use the formula $S = \frac{n(a + \ell)}{2}$ to find the sum of any arithmetic series.

Example 1 **Find the sum: 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9**

$a = 1, \ell = 9, n = 9$, so $S = \frac{9(1 + 9)}{2} = \frac{9 \cdot 10}{2} = 45$

Example 2 **Find the sum: −9 + (−5) + (−1) + 3 + 7 + 11 + 15**

$a = 29, \ell = 15, n = 7$, so $S = \frac{7(-9 + 15)}{2} = \frac{7 \cdot 6}{2} = 21$

Exercises

Find the sum of each arithmetic series.

1. 3 + 6 + 9 + 12 + 15 + 18 + 21 + 24

2. 10 + 15 + 20 + 25 + 30 + 35 + 40 + 45 + 50

3. −21 + (−16) + (−11) + (−6) + (−1) + 4 + 9 + 14

4. even whole numbers from 2 through 100

5. odd whole numbers between 0 and 100

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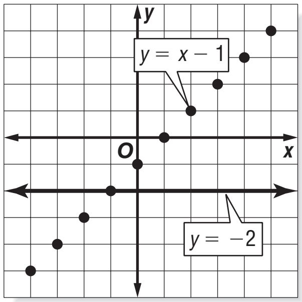
3-5 Enrichment

Taxicab Graphs

You have used a rectangular coordinate system to graph equations such as $y = x - 1$ on a coordinate plane. In a coordinate plane, the numbers in an ordered pair (x, y) can be any two real numbers.

A **taxicab plane** is different from the usual coordinate plane. The only points allowed are those that exist along the horizontal and vertical grid lines. You may think of the points as taxicabs that must stay on the streets.

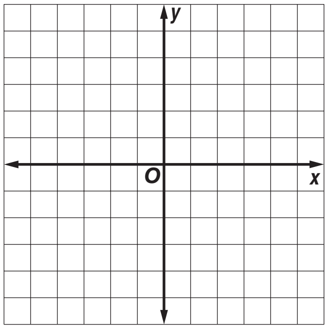
The taxicab graph shows the equations $y = -2$ and $y = x - 1$. Notice that one of the graphs is no longer a straight line. It is now a collection of separate points.



Graph these equations on the taxicab plane at the right.

1. $y = x + 1$ **2.** $y = -2x + 3$

3. $y = 2.5$ **4.** $x = -4$



Use your graphs for these problems.

5. Which of the equations has the same graph in both the usual coordinate plane and the taxicab plane?

6. Describe the form of equations that have the same graph in both the usual coordinate plane and the taxicab plane.

In the taxicab plane, distances are not measured diagonally, but along the streets. Write the taxi-distance between each pair of points.

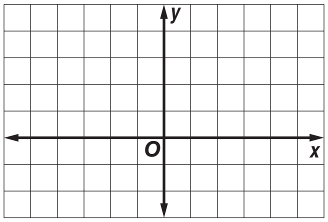
7. (0, 0) and (5, 2) **8.** (0, 0) and (-3, 2) **9.** (0, 0) and (2, 1.5)

10. (1, 2) and (4, 3) **11.** (2, 4) and (-1, 3) **12.** (0, 4) and (-2, 0)

Draw these graphs on the taxicab grid at the right.

13. The set of points whose taxi-distance from (0, 0) is 2 units.

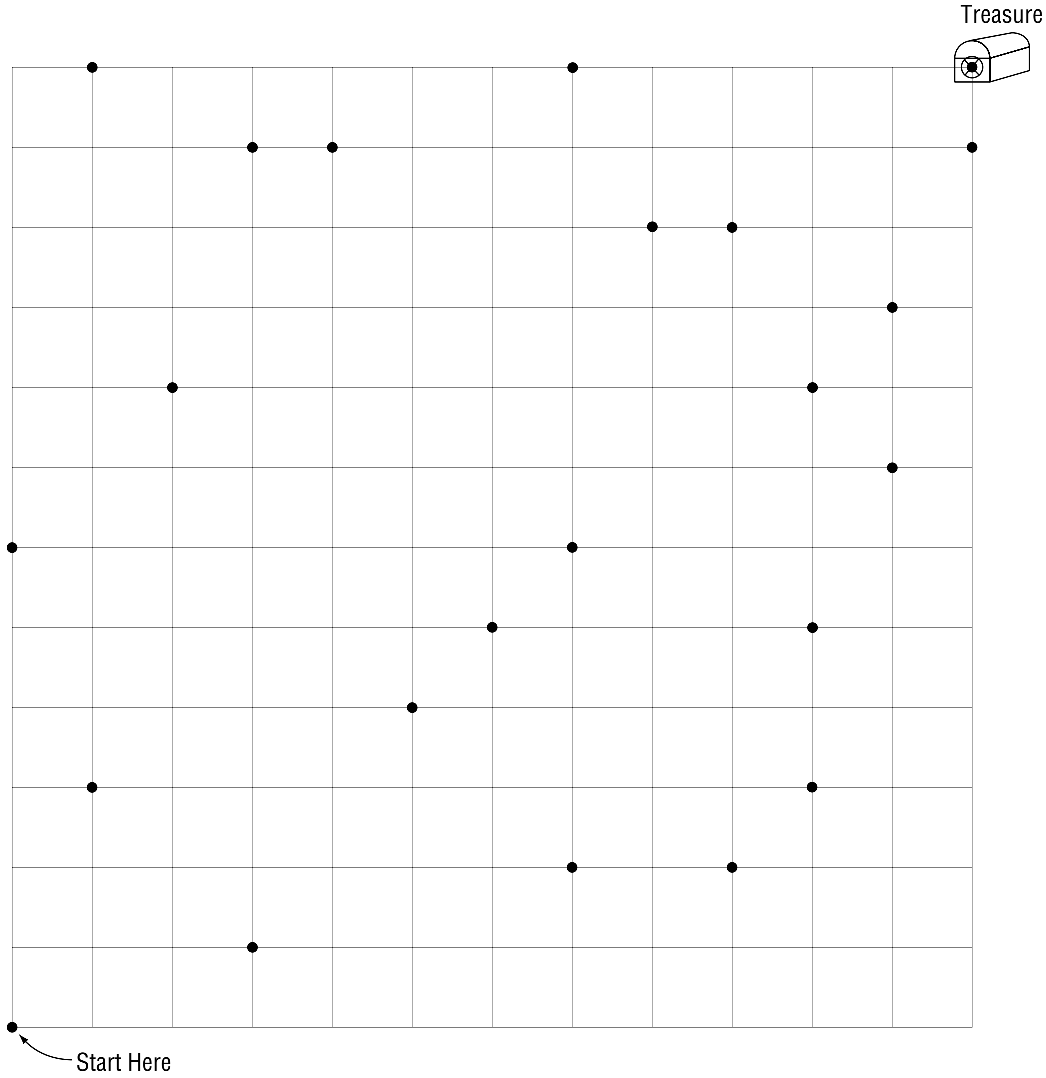
14. The set of points whose taxi-distance from (2, 1) is 3 units.



4-1 Enrichment

Treasure Hunt with Slopes

Using the definition of slope, draw lines with the slopes listed below. A correct solution will trace the route to the treasure.



1. 3	**2.** $\frac{1}{4}$	**3.** $-\frac{2}{5}$	**4.** 0
5. 1	**6.** -1	**7.** no slope	**8.** $\frac{2}{7}$
9. $\frac{3}{2}$	**10.** $\frac{1}{3}$	**11.** $-\frac{3}{4}$	**12.** 3

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4-2 Enrichment

nth Power Variation

An equation of the form $y = kx^n$, where $k \neq 0$, describes an nth power variation. The variable n can be replaced by 2 to indicate the second power of x (the square of x) or by 3 to indicate the third power of x (the cube of x).

Assume that the weight of a person of average build varies directly as the cube of that person's height. The equation of variation has the form $w = kh^3$.

The weight that a person's legs will support is proportional to the cross-sectional area of the leg bones. This area varies directly as the square of the person's height. The equation of variation has the form $s = kh^2$.

Answer each question.

1. For a person 6 feet tall who weighs 200 pounds, find a value for k in the equation $w = kh^3$.

2. Use your answer from Exercise 1 to predict the weight of a person who is 5 feet tall.

3. Find the value for k in the equation $w = kh^3$ for a baby who is 20 inches long and weighs 6 pounds.

4. How does your answer to Exercise 3 demonstrate that a baby is significantly fatter in proportion to its height than an adult?

5. For a person 6 feet tall who weighs 200 pounds, find a value for k in the equation $s = kh^2$.

6. For a baby who is 20 inches long and weighs 6 pounds, find an "infant value" for k in the equation $s = kh^2$.

7. According to the adult equation you found (Exercise 1), how much would an imaginary giant 20 feet tall weigh?

8. According to the adult equation for weight supported (Exercise 5), how much weight could a 20-foot tall giant's legs actually support?

9. What can you conclude from Exercises 7 and 8?

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4-3 Enrichment

Using Equations: Ideal Weight

You can find your ideal weight as follows.

A woman should weigh 100 pounds for the first 5 feet of height and 5 additional pounds for each inch over 5 feet (5 feet = 60 inches). A man should weigh 106 pounds for the first 5 feet of height and 6 additional pounds for each inch over 5 feet. These formulas apply to people with normal bone structures.

To determine your bone structure, wrap your thumb and index finger around the wrist of your other hand. If the thumb and finger just touch, you have normal bone structure. If they overlap, you are small-boned. If they don't overlap, you are large-boned. Small-boned people should decrease their calculated ideal weight by 10%. Large-boned people should increase the value by 10%.

Calculate the ideal weights of these people.

1. woman, 5 ft 4 in., normal-boned

2. man, 5 ft 11 in., large-boned

3. man, 6 ft 5 in., small-boned

4. you, if you are at least 5 ft tall

For Exercises 5–9, use the following information.

Suppose a normal-boned man is x inches tall. If he is at least 5 feet tall, then $x - 60$ represents the number of inches this man is over 5 feet tall. For each of these inches, his ideal weight is increased by 6 pounds. Thus, his proper weight (y) is given by the formula $y = 6(x - 60) + 106$ or $y = 6x - 254$. If the man is large-boned, the formula becomes $y = 6x - 254 + 0.10(6x - 254)$.

5. Write the formula for the weight of a large-boned man in slope-intercept form.

6. Derive the formula for the ideal weight (y) of a normal-boned female with height x inches. Write the formula in slope-intercept form.

7. Derive the formula in slope-intercept form for the ideal weight (y) of a large-boned female with height x inches.

8. Derive the formula in slope-intercept form for the ideal weight (y) of a small-boned male with height x inches.

9. Find the heights at which normal-boned males and large-boned females would weigh the same.

Lesson 4-3

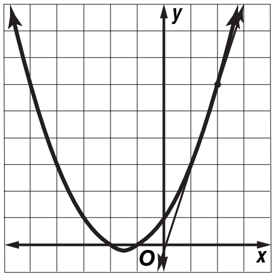
4-4 Enrichment

Tangent to a Curve

A tangent line is a line that intersects a curve at a point with the same rate of change, or slope, as the rate of change of the curve at that point.

For quadratic functions (functions of the form $ax^2 + bx + c$), the equation of the tangent line can be found. This is based on the fact that the slope through any two points on the curve is equal to the slope of the line tangent to the curve at the point whose x-value is halfway between the x-values of the other two points.

Example **To find the equation of a tangent line to the curve $y = x^2 + 3x + 2$ through the point (2, 12), first find two points on the curve whose x-values are equidistant from the x-value of the point the tangent needs to go through.**



Step 1: Find two more points. Use $x = 1$ and $x = 3$.
When $x = 1, y = 1^2 + 3(1) + 2$ or 6.
When $x = 3, y = 3^2 + 3(3) + 2$ or 20.
So, the two ordered pairs are (1, 6) and (3, 20).

Step 2: Find the slope of the line that goes through these two points.
$m = \frac{20 - 6}{3 - 1}$ or 7

Step 3: Now use this slope and the point (2, 12) to find the equation of the tangent line.

$y = mx + b$	Slope intercept form.
$12 = 7(2) + b$	Replace *x* with 2, *y* with 12, and *m* with 7.
$-2 = b$	Solve for *b*.

So, the equation of the tangent line to $y = x^2 + 3x + 2$ through the point (2, 12) is $y = 7x - 2$.

Exercises

For 1-3, find the equations of the lines tangent to each curve through the given point.

1. $y = x^2 - 3x+7$, (2, 5)

2. $y = 3x^2 + 4x - 5$, $(-4, 27)$

3. $y = 5 - x^2$, (1, 4)

4. Find the slope of the line tangent to the curve at $x = 0$ for the general equation $y = ax^2 + bx + c$.

5. Find the slope of the line tangent to the curve $y = ax^2 + bx + c$ at x by finding the slope of the line through the points $(0, c)$ and $(2x, 4ax^2 + 2bx + c)$. Does this answer work for $x = 0$ in the answer you found to problem 4?

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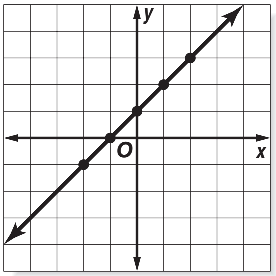
4-5 Enrichment

Collinearity

You have learned how to find the slope between two points on a line. Does it matter which two points you use? How does your choice of points affect the slope-intercept form of the equation of the line?

1. Choose three different pairs of points from the graph at the right. Write the slope-intercept form of the line using each pair.

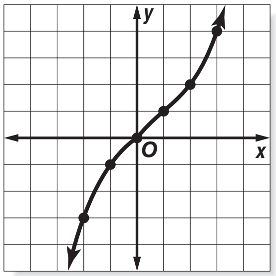


2. How are the equations related?

3. What conclusion can you draw from your answers to Exercises 1 and 2?

When points are contained in the same line, they are said to be **collinear**. Even though points may *look* like they form a straight line when connected, it does not mean that they actually do. By checking pairs of points on a line you can determine whether the line represents a linear relationship.

4. Choose several pairs of points from the graph at the right and write the slope-intercept form of the line using each pair.



5. What conclusion can you draw from your equations in Exercise 4? Is this a straight line?

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4-6 Enrichment

Latitude and Temperature

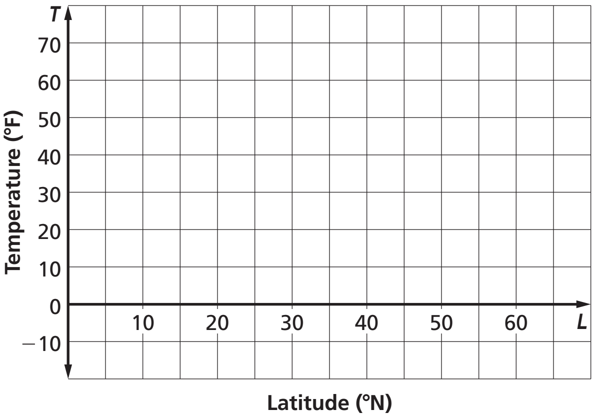
The *latitude* of a place on Earth is the measure of its distance from the equator. What do you think is the relationship between a city's latitude and its January temperature? At the right is a table containing the latitudes and January mean temperatures for fifteen U.S. cities.

U.S. City	Latitude	January Mean Temperature
Albany, New York	42:40 N	20.7°F
Albuquerque, New Mexico	35:07 N	34.3°F
Anchorage, Alaska	61:11 N	14.9°F
Birmingham, Alabama	33:32 N	41.7°F
Charleston, South Carolina	32:47 N	47.1°F
Chicago, Illinois	41:50 N	21.0°F
Columbus, Ohio	39:59 N	26.3°F
Duluth, Minnesota	46:47 N	7.0°F
Fairbanks, Alaska	64:50 N	−10.1°F
Galveston, Texas	29:14 N	52.9°F
Honolulu, Hawaii	21:19 N	72.9°F
Las Vegas, Nevada	36:12 N	45.1°F
Miami, Florida	25:47 N	67.3°F
Richmond, Virginia	37:32 N	35.8°F
Tucson, Arizona	32:12 N	51.3°F

Sources: www.indo.com and www.nws.noaa.gov/climatex.html

1. Use the information in the table to create a scatter plot and draw a line of best fit for the data.

2. Write an equation for the line of fit. Make a conjecture about the relationship between a city's latitude and its mean January temperature.



3. Use your equation to predict the January mean temperature of Juneau, Alaska, which has latitude 58:23 N.

4. What would you expect to be the latitude of a city with a January mean temperature of 15°F?

5. Was your conjecture about the relationship between latitude and temperature correct?

6. Research the latitudes and temperatures for cities in the southern hemisphere instead. Does your conjecture hold for these cities as well?

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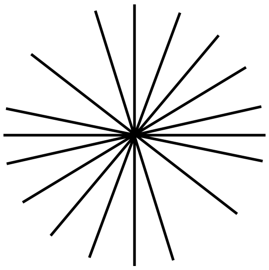
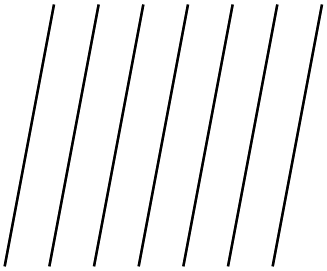
4-7 Enrichment

Pencils of Lines

All of the lines that pass through a single point in the same plane are called a **pencil of lines**.

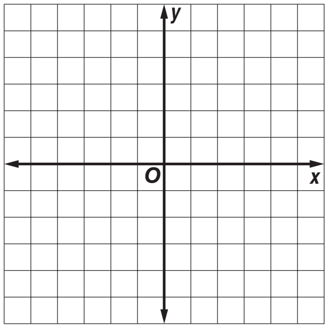
All lines with the same slope, but different intercepts, are also called a "pencil," a **pencil of parallel lines**.



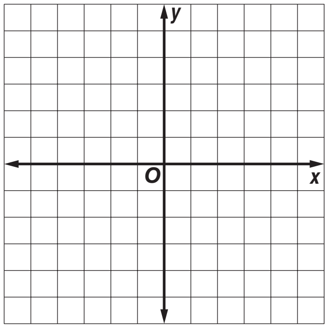


Graph some of the lines in each pencil.

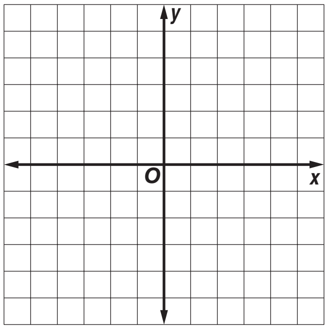
1. A pencil of lines through the point (1, 3)



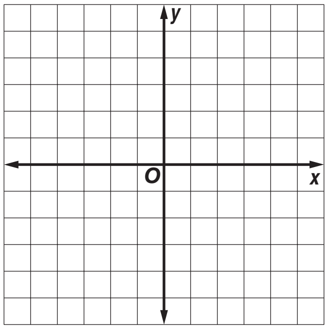
2. A pencil of lines described by $y - 4 = m(x - 2)$, where m is any real number



3. A pencil of lines parallel to the line $x - 2y = 7$



4. A pencil of lines described by $y = mx + 3m - 2$



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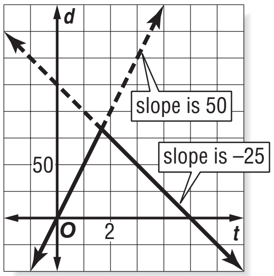
5-1 Enrichment

Graphing a Trip



The distance formula, $d = rt$, is used to solve many types of problems. If you graph an equation such as $d = 50t$, the graph is a model for a car going at 50 mi/h. The time the car travels is t; the distance in miles the car covers is d. The slope of the line is the speed.

Suppose you drive to a nearby town and return. You average 50 mi/h on the trip out but only 25 mi/h on the trip home. The round trip takes 5 hours. How far away is the town?

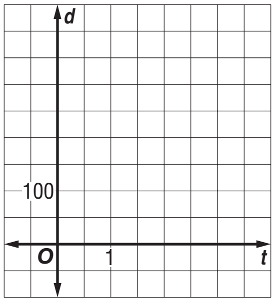
The graph at the right represents your trip. Notice that the return trip is shown with a negative slope because you are driving in the opposite direction.

Lesson 5-1

Solve each problem.

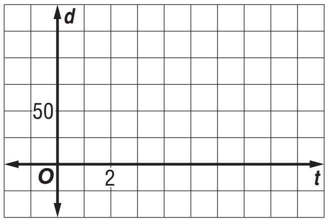
1. Estimate the answer to the problem in the above example. About how far away is the town?

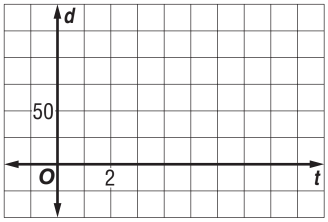
2. Graph this trip and solve the problem. An airplane has enough fuel for 3 hours of safe flying. On the trip out the pilot averages 200 mi/h flying against a headwind. On the trip back, the pilot averages 250 mi/h. How long a trip out can the pilot make?



3. Graph this trip and solve the problem. You drive to a town 100 miles away. On the trip out you average 25 mi/h. On the trip back you average 50 mi/h. How many hours do you spend driving?

4. Graph this trip and solve the problem. You drive at an average speed of 50 mi/h to a discount shopping plaza, spend 2 hours shopping, and then return at an average speed of 25 mi/h. The entire trip takes 8 hours. How far away is the shopping plaza?





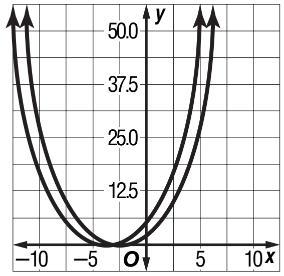
5-2 Enrichment

Intersection of Two Parabolas

Substitution can be used to find the intersection of two parabolas. Replace the *y*-value in one of the equations with the *y*-value in terms of *x* from the other equation.

Example **Find the intersection of the two parabolas.**
$y = x^2 + 5x + 6$
$y = x^2 + 4x + 3$

Graph the equations.



From the graph, notice that the two graphs intersect in one point.

Use substitution to solve for the point of intersection.

$x^2 + 5x + 6 = x^2 + 4x + 3$

$5x + 6 = 4x + 3$ Subtract x^2 from both sides.

$x + 6 = 3$ Subtract $4x$ from both sides.

$x = -3$ Subtract 6 from both sides.

So, the graphs intersect at $x = -3$.

Replace x with -3 in either equation to find the y-value.

$y = x^2 + 5x + 6$ Original equation

$y = (-3)2 + 5(-3) + 6$ Replace x with -3.

$y = 9 - 15 + 6$ *or* 0. Simplify.

So, the point of intersection is $(-3, 0)$

Exercises

Use substitution to find the point of intersection of the graphs of each pair of equations.

1. $y = x^2 + 8x + 7$
   $y = x^2 + 2x + 1$

2. $y = x^2 + 6x + 8$
   $y = x^2 + 4x + 4$

3. $y = x^2 + 5x + 6$
   $y = x^2 + 7x + 6$

5-3 Enrichment

Solving Systems of Equations in Three Variables

Systems of equations can involve more than 2 equations and 2 variables. It is possible to solve a system of 3 equations and 3 variables using elimination.

Example **Solve the following system.**

$$\begin{aligned} x + y + z &= 6 \\ 3x - y + z &= 8 \\ x - z &= 2 \end{aligned}$$

Step 1: Use elimination to get rid of the y in the first two equations.

$$\begin{array}{r} x + y + z = 6 \\ \underline{3x - y + z = 8} \\ 4x + 2z = 14 \end{array}$$

Step 2: Use the equation you found in step 1 and the third equation to eliminate the z.

$$\begin{array}{r} 4x + 2z = 14 \\ x - z = 2 \end{array}$$

Multiply the second equation by 2 so that the z's will eliminate.

$$\begin{array}{r} 4x + 2z = 14 \\ \underline{2x - 2z = 4} \\ 6x = 18 \end{array}$$

So, $x = 3$.

Step 3: Replace x with 3 in the third original equation to determine z.

$3 - z = 2$, *so* $z = 1$.

Step 4: Replace x with 3 and z with 1 in either of the first two original equation to determine the value of y.

$3 + y + 1 = 6$ or $4 + y = 6$. So, $y = 2$.

So, the solution to the system of equations is (3, 2, 1).

Exercises **Solve each system of equations.**

1. $\begin{aligned} 3x + 2y + z &= 42 \\ 2y + z + 12 &= 3x \\ x - 3y &= 0 \end{aligned}$

2. $\begin{aligned} x + y + z &= -3 \\ 2x + 3y + 5z &= -4 \\ 2y - z &= 4 \end{aligned}$

3. $\begin{aligned} x + y + z &= 7 \\ x + 2y + z &= 10 \\ 2y + z &= 5 \end{aligned}$

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5-4 Enrichment

George Washington Carver and Percy Julian

In 1990, George Washington Carver and Percy Julian became the first African Americans elected to the National Inventors Hall of Fame. Carver (1864–1943) was an agricultural scientist known worldwide for developing hundreds of uses for the peanut and the sweet potato. His work revitalized the economy of the southern United States because it was no longer dependent solely upon cotton. Julian (1898–1975) was a research chemist who became famous for inventing a method of making a synthetic cortisone from soybeans. His discovery has had many medical applications, particularly in the treatment of arthritis.

There are dozens of other African American inventors whose accomplishments are not as well known. Their inventions range from common household items like the ironing board to complex devices that have revolutionized manufacturing. The exercises that follow will help you identify just a few of these inventors and their inventions.

Match the inventors with their inventions by matching each system with its solution. (Not all the solutions will be used.)

1.	Sara Boone	$x + y = 2$ $x - y = 10$	**A.**	(1, 4)	automatic traffic signal
2.	Sarah Goode	$x = 2 - y$ $2y + x = 9$	**B.**	(4, −2)	eggbeater
3.	Frederick M. Jones	$y = 2x + 6$ $y = -x - 3$	**C.**	(−2, 3)	fire extinguisher
4.	J. L. Love	$2x + 3y = 8$ $2x - y = -8$	**D.**	(−5, 7)	folding cabinet bed
5.	T. J. Marshall	$y - 3x = 9$ $2y + x = 4$	**E.**	(6, −4)	ironing board
6.	Jan Matzeliger	$y + 4 = 2x$ $6x - 3y = 12$	**F.**	(−2, 4)	pencil sharpener
7.	Garrett A. Morgan	$3x - 2y = -5$ $3y - 4x = 8$	**G.**	(−3, 0)	portable X-ray machine
8.	Norbert Rillieux	$3x - y = 12$ $y - 3x = 15$	**H.**	(2, −3)	player piano
			I.	no solution	evaporating pan for refining sugar
			J.	infinitely many solutions	lasting (shaping) machine for manufacturing shoes

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5-5 Enrichment

Cramer's Rule

Cramer's Rule is a method for solving a system of equations. To use Cramer's Rule, set up a matrix to represent the equations. A matrix is a way of organizing data.

Example **Solve the following system of equations using Cramer's Rule.**

$$2x + 3y = 13$$
$$x + y = 5$$

Step 1: Set up a matrix representing the coefficients of x and y.

$$A = \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \end{array}$$

Step 2: Find the determinant of matrix A.

If a matrix $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, then the determinant, $\det(A) = ad - bc$.

$\det(A) = 2(1) - 1(3) = -1$

Step 3: Replace the first column in A with 13 and 5 and find the determinant of the new matrix.

$A_1 = \begin{vmatrix} 13 & 3 \\ 5 & 1 \end{vmatrix}$; $\det(A_1) = 13(1) - 5(3) = -2$

Step 4: To find the value of x in the solution to the system of equations, determine the value of $\frac{\det(A_1)}{\det(A)}$.

$\frac{\det(A_1)}{\det(A)} = \frac{-2}{-1}$ or 2

Step 5: Repeat the process to find the value of y. This time, replace the second column with 13 and 5 and find the determinant.

$A_2 = \begin{vmatrix} 2 & 13 \\ 1 & 5 \end{vmatrix}$; $\det(A_2) = 2(5) - 1(13) = -3$ and $\frac{\det(A_2)}{\det(A)} = \frac{-3}{-1}$ or 3.

So, the solution to the system of equations is (2, 3).

Exercises **Use Cramer's Rule to solve each system of equations.**

1. $2x + y = 1$
$3x + 5y = 5$

2. $x + y = 4$
$2x - 3y = -2$

3. $x - y = 4$
$3x - 5y = 8$

4. $4x - y = 3$
$x + y = 7$

5. $3x - 2y = 7$
$2x + y = 14$

6. $6x - 5y = 1$
$3x + 2y = 5$

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6-1 Enrichment

Triangle Inequalities

Recall that a line segment can be named by the letters of its endpoints. Line segment *AB* (written as $\overline{AB}$) has points *A* and *B* for endpoints. The *length* of *AB* is written without the bar as *AB*.

$$AB > BC \qquad m\angle A < m\angle B$$

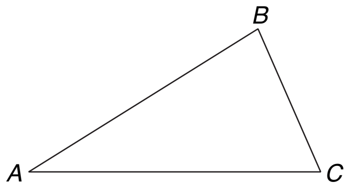
The statement on the left above shows that $\overline{AB}$ is shorter than $\overline{BC}$. The statement on the right above shows that the measure of angle *A* is less than that of angle *B*.

These three inequalities are true for any triangle *ABC*, no matter how long the sides.

a. $AB + BC > AC$

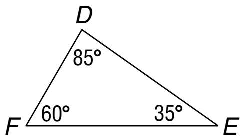
b. If $AB > AC$, then $m\angle C > m\angle B$.

c. If $m\angle C > m\angle B$, then $AB > AC$.

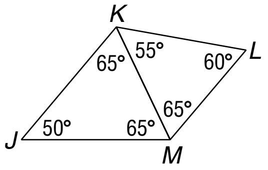


Use the three triangle inequalities for these problems.

1. List the sides of triangle *DEF* in order of increasing length.



2. In the figure at the right, which line segment is the shortest?

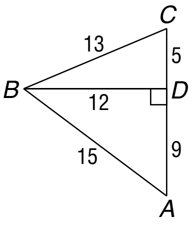


3. Explain why the lengths 5 cm, 10 cm, and 20 cm could not be used to make a triangle.

4. Two sides of a triangle measure 3 in. and 7 in. Between which two values must the third side be?

5. In triangle *XYZ*, $XY = 15$, $YZ = 12$, and $XZ = 9$. Which is the greatest angle? Which is the least?

6. List the angles $\angle A$, $\angle C$, $\angle ABC$, and $\angle ABD$, in order of increasing size.



6-2 Enrichment

Quadratic Inequalities

Like linear inequalities, inequalities with higher degrees can also be solved. Quadratic Inequalities have a degree of 2. The following example shows how to solve quadratic inequalities.

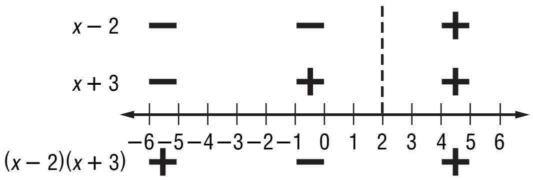
Example **Solve $(x + 3)(x - 2) > 0$.**

Step 1 Determine what values of x will make the left side 0. In other words, what values of x will make either $x + 3 = 0$ or $x - 2 = 0$?

$x = -3$ or 2

Step 2 Plot these points on a number line. Above the number line, place a + if $x + 3$ is positive for that region or a − if $x + 3$ is negative for that region. Next, above the signs you have just entered; do the same for $x - 2$.

Step 3 Below the chart, enter the product of the two signs. Your sign chart should look like the following:



The final positive regions correspond to values for which the quadratic expression is greater than 0. So, the answer is

$x < -3$ or $x > 2$.

Exercises

Solve each inequality.

1. $(x - 1)(x + 2) > 0$

2. $(x + 5)(x + 2) > 0$

3. $(x - 1)(x - 5) < 0$

4. $(x + 2)(x - 4) \leq 0$

5. $(x - 3)(x + 2) \geq 0$

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6-3 Enrichment

Carlos Montezuma

During his lifetime, Carlos Montezuma (1865?–1923) was one of the most influential Native Americans in the United States. He was recognized as a prominent physician and was also a passionate advocate of the rights of Native American peoples. The exercises that follow will help you learn some interesting facts about Dr. Montezuma's life.

Solve each inequality. The word or phrase next to the equivalent inequality will complete the statement correctly.

1. $-2k > 10$
Montezuma was born in the state of __?__.

a. $k < -5$ Arizona
b. $k > -5$ Montana
c. $k > 12$ Utah

2. $5 \geq r - 9$
He was a Native American of the Yavapais, who are a __?__ people.

a. $r \leq -4$ Navajo
b. $r \geq -4$ Mohawk
c. $r \leq 14$ Mohave-Apache

3. $-y \leq -9$
Montezuma received a medical degree from __?__ in 1889.

a. $y \geq 9$ Chicago Medical College
b. $y \geq -9$ Harvard Medical School
c. $y \leq 9$ Johns Hopkins University

4. $-3 + q > 12$
As a physician, Montezuma's field of specialization was __?__.

a. $q > -4$ heart surgery
b. $q > 15$ internal medicine
c. $q < -15$ respiratory diseases

5. $5 + 4x - 14 \leq x$
For much of his career, he maintained a medical practice in __?__.

a. $x \leq 9$ New York City
b. $x \leq 3$ Chicago
c. $x \geq -9$ Boston

6. $7 - t < 7 + t$
In addition to maintaining his medical practice, he was also a(n) __?__.

a. $t > 7$ director of a blood bank
b. $t > 0$ instructor at a medical college
c. $t < -7$ legal counsel to physicians

7. $3a + 8 \geq 4a - 10$
Montezuma founded, wrote, and edited __?__, a monthly newsletter that addressed Native American concerns.

a. $a \leq -2$ Yavapai
b. $a \geq 18$ Apache
c. $a \leq 18$ Wassaja

8. $6n > 8n - 12$
Montezuma testified before a committee of the United States Congress concerning his work in treating __?__.

a. $n < 6$ appendicitis
b. $n > -6$ asthma
c. $n > -10$ heart attacks

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6-4 Enrichment

Some Properties of Inequalities

The two expressions on either side of an inequality symbol are sometimes called the *first* and *second* members of the inequality.

If the inequality symbols of two inequalities point in the same direction, the inequalities have the same sense. For example, $a < b$ and $c < d$ have the same sense; $a < b$ and $c > d$ have opposite senses.

In the problems on this page, you will explore some properties of inequalities.

Three of the four statements below are true for all numbers *a* and *b* (or *a*, *b*, *c*, and *d*). Write each statement in algebraic form. If the statement is true for all numbers, prove it. If it is not true, give an example to show that it is false.

1. Given an inequality, a new and equivalent inequality can be created by interchanging the members and reversing the sense.

2. Given an inequality, a new and equivalent inequality can be created by changing the signs of both terms and reversing the sense.

3. Given two inequalities with the same sense, the sum of the corresponding members are members of an equivalent inequality with the same sense.

4. Given two inequalities with the same sense, the difference of the corresponding members are members of an equivalent inequality with the same sense.

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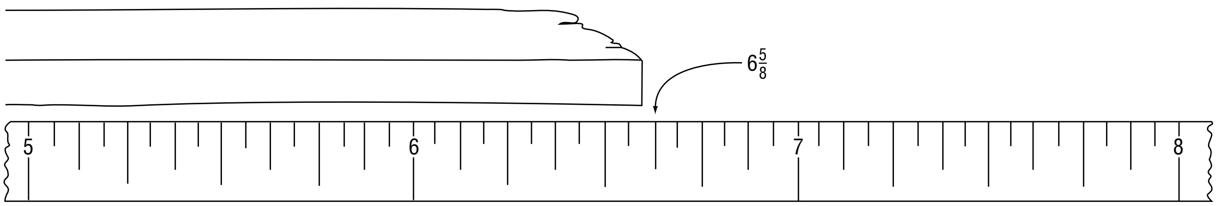
6-5 Enrichment

Precision of Measurement

The precision of a measurement depends both on your accuracy in measuring and the number of divisions on the ruler you use. Suppose you measured a length of wood to the nearest one-eighth of an inch and got a length of $6\frac{5}{8}$ in.



The drawing shows that the actual measurement lies somewhere between $6\ \frac{9}{16}$ in. and $6\frac{11}{16}$ in. This measurement can be written using the symbol ±, which is read *plus or minus*. It can also be written as a compound inequality.

$$6\frac{5}{8} \pm \frac{1}{16} \text{ in.} \qquad 6\frac{9}{16} \text{ in.} \leq m \leq 6\frac{11}{16} \text{ in.}$$

In this example, $\frac{1}{16}$ in. is the absolute error. The absolute error is one-half the smallest unit used in a measurement.

Write each measurement as a compound inequality. Use the variable m.

1. $3\frac{1}{2} \pm \frac{1}{4}$ in.

2. 9.78 ± 0.005 cm

3. 2.4 ± 0.05 g

4. $28 \pm \frac{1}{2}$ ft

5. 15 ± 0.5 cm

6. $\frac{11}{16} \pm \frac{1}{64}$ in.

For each measurement, give the smallest unit used and the absolute error.

7. $12.5 \text{ cm} \leq m \leq 13.5 \text{ cm}$

8. $12\frac{1}{8} \text{ in.} \leq m \leq 12\frac{3}{8} \text{ in.}$

9. $56\frac{1}{2} \text{ in.} \leq m \leq 57\frac{1}{2} \text{ in.}$

10. $23.05 \text{ mm} \leq m \leq 23.15 \text{ mm}$

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6-6 Enrichment

Linear Programming

Linear programming can be used to maximize or minimize costs. It involves graphing a set of linear inequalities and using the region of intersection. You will use linear programming to solve the following problem.

Example 1 **Layne's Gift Shoppe sells at most 500 items per week. To meet her customers' demands, she sells at least 100 stuffed animals and 75 greeting cards. If the profit for each stuffed animal is \$2.50 and the profit for each greeting card is \$1.00, the equation $P(a, g) = 2.50a + 1.00g$ can be used to represent the profit. How many of each should she sell to maximize her profit?**

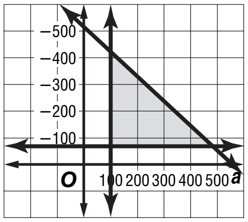
Write the inequalities:

$a + g \leq 500$

$a \geq 100$

$g \geq 75$

Graph the inequalities:



Find the vertices of the triangle formed: (100, 75), (100, 400), and (425, 75) Substitute the values of the vertices into the equation found above:

$$2.50(100) + 1(75) = 325$$

$$2.50(100) + 1(400) = 650$$

$$2.50(425) + 1(75) = 1137.50$$

So, the maximum profit is \$1137.50.

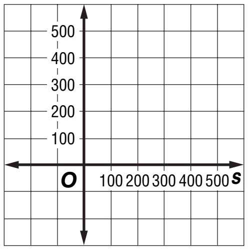
Exercises

The Spirit Club is selling shirts and banners. They sell at most 400 of the two items. To meet the demands of the students, they must sell at least 50 shirts and 100 banners. The profit on each shirt is \$4.00 and the profit on each banner is \$1.50, the equation $P(s, b) = 4.00s + 1.50b$ can be used to represent the profit. How many should they sell of each to maximize the profit?

1. Write the inequalities to represent this situation.

2. Graph the inequalities from part 1.

3. Find the vertices of the figure formed.



4. What is the maximum profit the Spirit Club can make?

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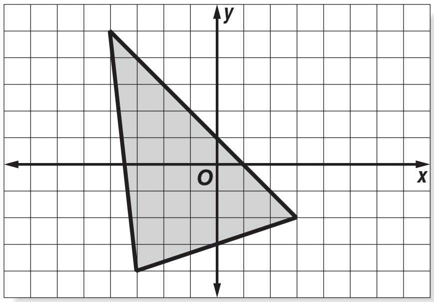
6-7 Enrichment

Describing Regions

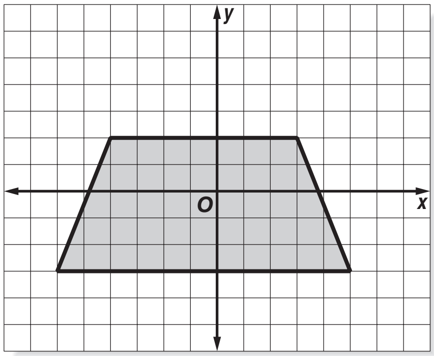
The shaded region inside the triangle can be described with a system of three inequalities.

$$y < -x + 1$$
$$y > \frac{1}{3}x - 3$$
$$y > -9x - 31$$

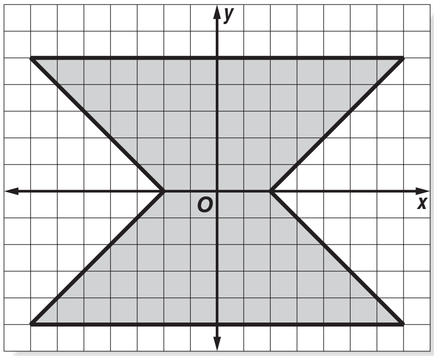


Write systems of inequalities to describe each region. You may first need to divide a region into triangles or quadrilaterals.

1.



2.



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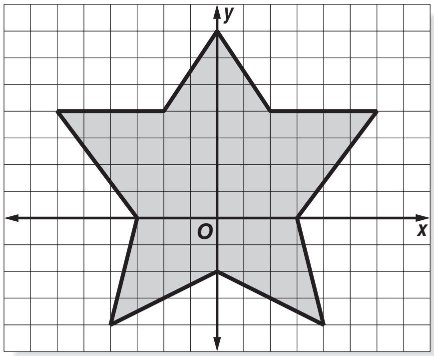

NAME ____ DATE______________ PERIOD _____

7-1 Enrichment

An Wang

An Wang (1920–1990) was an Asian-American who became one of the pioneers of the computer industry in the United States. He grew up in Shanghai, China, but came to the United States to further his studies in science. In 1948, he invented a magnetic pulse controlling device that vastly increased the storage capacity of computers. He later founded his own company, Wang Laboratories, and became a leader in the development of desktop calculators and word processing systems. In 1988, Wang was elected to the National Inventors Hall of Fame.

Digital computers store information as numbers. Because the electronic circuits of a computer can exist in only one of two states, open or closed, the numbers that are stored can consist of only two digits, 0 or 1. Numbers written using only these two digits are called **binary numbers**. To find the decimal value of a binary number, you use the digits to write a *polynomial in 2*. For instance, this is how to find the decimal value of the number 1001101_2. (The subscript 2 indicates that this is a binary number.)

$$\begin{aligned} 1001101_2 &= 1 \times \underline{2^6} + 0 \times \underline{2^5} + 0 \times \underline{2^4} + 1 \times \underline{2^3} + 1 \times \underline{2^2} + 0 \times \underline{2^1} + 1 \times \underline{2^0} \\ &= 1 \times \underline{64} + 0 \times \underline{32} + 0 \times \underline{16} + 1 \times \underline{8} + 1 \times \underline{4} + 0 \times \underline{2} + 1 \times \underline{1} \\ &= 64 + 0 + 0 + 8 + 4 + 0 + 1 \\ &= 77 \end{aligned}$$

Find the decimal value of each binary number.

1. 1111_2

2. 10000_2

3. 11000011_2

4. 10111001_2

Write each decimal number as a binary number.

5. 8

6. 11

7. 29

8. 117

9. The chart at the right shows a set of decimal code numbers that is used widely in storing letters of the alphabet in a computer's memory. Find the code numbers for the letters of your name. Then write the code for your name using binary numbers.

The American Standard Guide for Information Interchange (ASCII)

A	65	**N**	78	**a**	97	**n**	110
B	66	**O**	79	**b**	98	**o**	111
C	67	**P**	80	**c**	99	**p**	112
D	68	**Q**	81	**d**	100	**q**	113
E	69	**R**	82	**e**	101	**r**	114
F	70	**S**	83	**f**	102	**s**	115
G	71	**T**	84	**g**	103	**t**	116
H	72	**U**	85	**h**	104	**u**	117
I	73	**V**	86	**i**	105	**v**	118
J	74	**W**	87	**j**	106	**w**	119
K	75	**X**	88	**k**	107	**x**	120
L	76	**Y**	89	**l**	108	**y**	121
M	77	**Z**	90	**m**	109	**z**	122

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7-2 Enrichment

Patterns with Powers

Use your calculator, if necessary, to complete each pattern.

a. 2^{10} = ____________

2^9 = ____________

2^8 = ____________

2^7 = ____________

2^6 = ____________

2^5 = ____________

2^4 = ____________

2^3 = ____________

2^2 = ____________

2^1 = ____________

b. 5^{10} = ____________

5^9 = ____________

5^8 = ____________

5^7 = ____________

5^6 = ____________

5^5 = ____________

5^4 = ____________

5^3 = ____________

5^2 = ____________

5^1 = ____________

c. 4^{10} = ____________

4^9 = ____________

4^8 = ____________

4^7 = ____________

4^6 = ____________

4^5 = ____________

4^4 = ____________

4^3 = ____________

4^2 = ____________

4^1 = ____________

Study the patterns for a, b, and c above. Then answer the questions.

1. Describe the pattern of the exponents from the top of each column to the bottom.

2. Describe the pattern of the powers from the top of the column to the bottom.

3. What would you expect the following powers to be?
2^0 5^0 4^0

4. Refer to Exercise 3. Write a rule. Test it on patterns that you obtain using 22, 25, and 24 as bases.

Study the pattern below. Then answer the questions.

$0^3 = 0$ $0^2 = 0$ $0^1 = 0$ $0^0 =$ __?__ 0^{-1} does not exist. 0^{-2} does not exist. 0^{-3} does not exist.

5. Why do 0^{-1}, 0^{-2}, and 0^{-3} not exist?

6. Based upon the pattern, can you determine whether 0^0 exists?

7. The symbol 0^0 is called an **indeterminate**, which means that it has no unique value. Thus it does not exist as a unique real number. Why do you think that 0^0 cannot equal 1?

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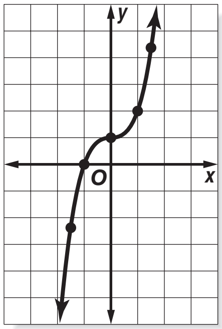
7-3 Enrichment

Polynomial Functions

Suppose a linear equation such as $-3x + y = 4$ is solved for y. Then an equivalent equation, $y = 3x + 4$, is found. Expressed in this way, y is a function of x, or $f(x) = 3x + 4$. Notice that the right side of the equation is a binomial of degree 1.

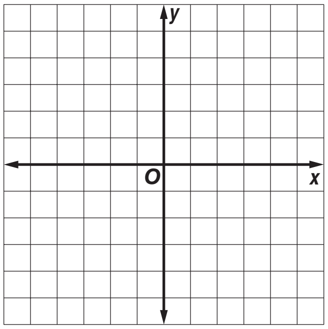
Higher-degree polynomials in x may also form functions. An example is $f(x) = x^3 + 1$, which is a polynomial function of degree 3. You can graph this function using a table of ordered pairs, as shown at the right.

x	y
$-1\frac{1}{2}$	$-2\frac{3}{8}$
-1	0
0	1
1	2
$1\frac{1}{2}$	$4\frac{3}{8}$

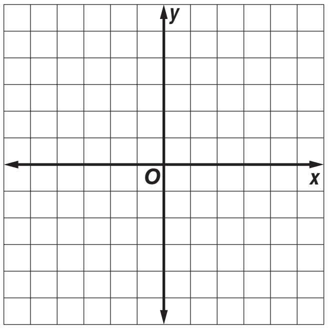


For each of the following polynomial functions, make a table of values for x and $y = f(x)$. Then draw the graph on the grid.

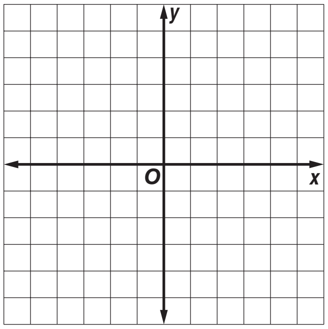
1. $f(x) = 1 - x^2$



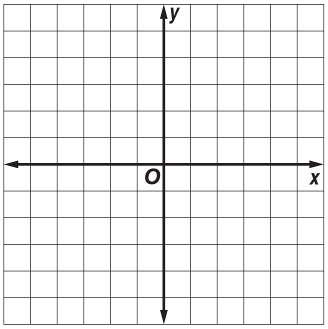
2. $f(x) = x^2 - 5$



3. $f(x) = x^2 + 4x - 1$



4. $f(x) = x^3$



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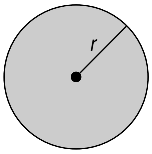
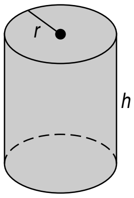
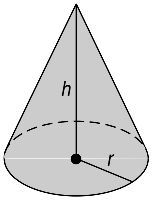
7-4 Enrichment

Circular Areas and Volumes

Area of Circle	**Volume of Cylinder**	**Volume of Cone**
$A = \pi r^2$	$V = \pi r^2 h$	$V = \frac{1}{3}\pi r^2 h$

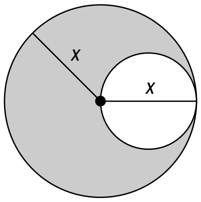






Write an algebraic expression for each shaded area. (Recall that the diameter of a circle is twice its radius.)

1.



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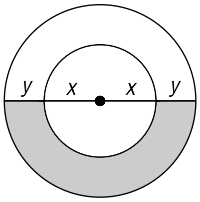

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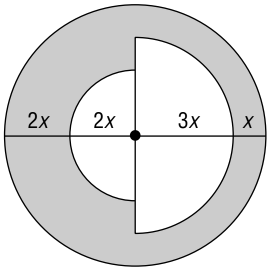
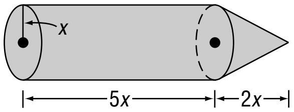


Write an algebraic expression for the total volume of each figure.

4.



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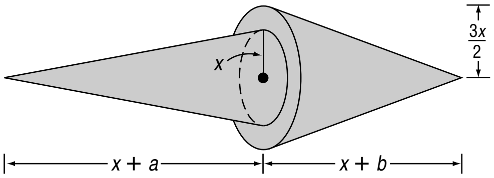
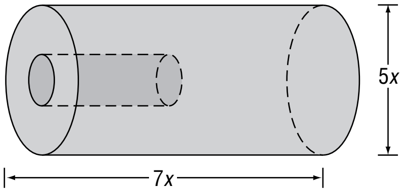
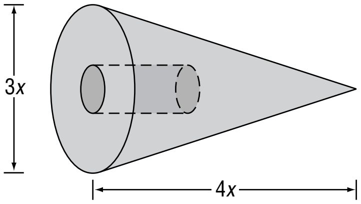


Each figure has a cylindrical hole with a radius of 2 inches and a height of 5 inches. Find each volume.

6.



7.



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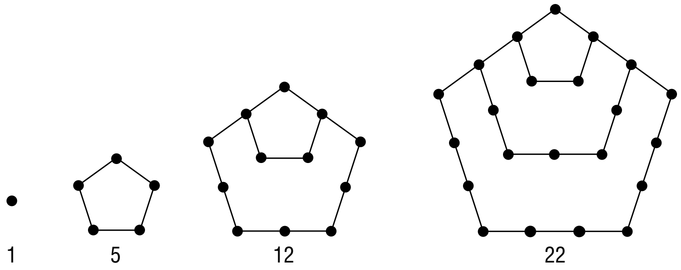
7-5 Enrichment

Figurate Numbers

The numbers below are called **pentagonal numbers**. They are the numbers of dots or disks that can be arranged as pentagons.



1. Find the product $\frac{1}{2}n(3n - 1)$.

2. Evaluate the product in Exercise 1 for values of n from 1 through 4.

3. What do you notice?

4. Find the next six pentagonal numbers.

5. Find the product $\frac{1}{2}n(n + 1)$.

6. Evaluate the product in Exercise 5 for values of n from 1 through 5. On another sheet of paper, make drawings to show why these numbers are called the triangular numbers.

7. Find the product $n(2n - 1)$.

8. Evaluate the product in Exercise 7 for values of n from 1 through 5. Draw these hexagonal numbers.

9. Find the first 5 square numbers. Also, write the general expression for any square number.

The numbers you have explored above are called the plane figurate numbers because they can be arranged to make geometric figures. You can also create solid figurate numbers.

10. If you pile 10 oranges into a pyramid with a triangle as a base, you get one of the tetrahedral numbers. How many layers are there in the pyramid? How many oranges are there in the bottom layers?

11. Evaluate the expression $\frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$ for values of n from 1 through 5 to find the first five tetrahedral numbers.

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7-6 Enrichment

Pascal's Triangle

This arrangement of numbers is called Pascal's Triangle. It was first published in 1665, but was known hundreds of years earlier.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

1. Each number in the triangle is found by adding two numbers. What two numbers were added to get the 6 in the 5th row?

2. Describe how to create the 6th row of Pascal's Triangle.

3. Write the numbers for rows 6 through 10 of the triangle.

Row 6:
Row 7:
Row 8:
Row 9:
Row 10:

Multiply to find the expanded form of each product.

4. $(a + b)^2$

5. $(a + b)^3$

6. $(a + b)^4$

Now compare the coefficients of the three products in Exercises 4–6 with Pascal's Triangle.

7. Describe the relationship between the expanded form of $(a + b)^n$ and Pascal's Triangle.

8. Use Pascal's Triangle to write the expanded form of $(a + b)^6$.

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7-7 Enrichment

Sums and Differences of Cubes

Recall the formulas for finding some special products:

Perfect-square trinomials: $(a + b)^2 = a^2 + 2ab + b^2$ or
$(a - b)^2 = a^2 - 2ab + b^2$

Difference of two squares: $(a + b)(a - b) = a^2 - b^2$

A pattern also exists for finding the cube of a sum $(a + b)^3$.

1. Find the product of $(a + b)(a + b)(a + b)$.

2. Use the pattern from Exercise 1 to evaluate $(x + 2)^3$.

3. Based on your answer to Exercise 1, predict the pattern for the cube of a difference $(a - b)^3$.

4. Find the product of $(a - b)(a - b)(a - b)$ and compare it to your answer for Exercise 3.

5. Use the pattern from Exercise 4 to evaluate $(x - 4)^3$.

Find each product.

6. $(x + 6)^3$

7. $(x - 10)^3$

8. $(3x - y)^3$

9. $(2x - y)^3$

10. $(4x + 3y)^3$

11. $(5x + 2)^3$

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8-1 Enrichment

Finding the GCF by Euclid's Algorithm

Finding the greatest common factor of two large numbers can take a long time using prime factorizations. This method can be avoided by using Euclid's Algorithm as shown in the following example.

Example **Find the GCF of 209 and 532.**

Divide the greater number, 532, by the lesser, 209.

Divide the remainder into the divisor above. Repeat this process until the remainder is zero. The last nonzero remainder is the GCF.

$$\begin{array}{rrrrr}
 & 2 & & & \\
209 & \overline{)532} & & & \\
 & \underline{418} & 1 & & \\
 & 114 & \overline{)209} & & \\
 & & \underline{114} & 1 & \\
 & & 95 & \overline{)114} & \\
 & & & \underline{95} & 5 \\
 & & & \mathbf{19} & \overline{)95} \\
 & & & & \underline{95} \\
 & & & & 0
\end{array}$$

The divisor, 19, is the GCF of 209 and 532.

Suppose the GCF of two numbers is found to be 1. Then the numbers are said to be **relatively prime**.

Find the GCF of each group of numbers by using Euclid's Algorithm.

1. 187; 578

2. 1802; 106

3. 161; 943

4. 215; 1849

5. 1325; 3498

6. 3484; 5963

7. 33,583; 4257

8. 453; 484

9. 95; 209; 589

10. 518; 407; 851

11. $17a^2x^2z$; $1615axz^2$

12. $752cf^3$; $893c^3f^3$

13. $979r^2s^2$; $495rs^3$, $154r^3s^3$

14. $360x^5y^7$; $328xy$; $568x^3y^3$

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8-2 Enrichment

Linear Combinations

The greatest common factor, GCF, of two numbers can be written as a linear combination of the two numbers. A linear combination is an expression of the form $Ax + By$.

Example **Write the greatest common factor of 52 and 36 as a linear combination.**

First, use the Euclidean Algorithm to find the greatest common factor of the two numbers.

$$\begin{array}{r} 1 \\ 36\overline{)52} \\ \underline{36} \end{array}$$

Divide the greater number by the lesser number.

$$\begin{array}{r} 2 \\ 16\overline{)36} \\ \underline{32} \end{array} \quad \text{original divisor;}$$

Then divide using the remainder as the new divisor.

$$\begin{array}{r} 4 \\ 4\overline{)16} \\ \underline{16} \\ 0 \end{array} \quad \text{second divisor;}$$

Divide again.

Stop dividing.

Last divisor used is the GCF. In this case, 4 is the GCF for 36 and 52.

To write 4 as a linear combination of 36 and 52, it needs to be written as:

$4 = 36x + 52y$, where x and y are some integers.

Use trial and error to determine the two integers.

The two integers that work are $x = 3$ and $y = -2$. So, the linear combination for the greatest common factor of 52 and 36 is:

$4 = 36(3) + 52(-2)$

Exercises

Write the greatest common factor for each pair of numbers as a linear combination.

1. 16, 64

2. 21, 28

3. 3, 18

4. 15, 36

5. 6, 8

6. 18, 42

8-3 Enrichment

Puzzling Primes

A prime number has only two factors, itself and 1. The number 6 is not prime because it has 2 and 3 as factors; 5 and 7 are prime. The number 1 is not considered to be prime.

1. Use a calculator to help you find the 25 prime numbers less than 100.

Prime numbers have interested mathematicians for centuries. They have tried to find expressions that will give all the prime numbers, or only prime numbers. In the 1700s, Euler discovered that the trinomial $x^2 + x + 41$ will yield prime numbers for values of x from 0 through 39.

2. Find the prime numbers generated by Euler's formula for x from 0 through 7.

3. Show that the trinomial $x^2 + x + 31$ will not give prime numbers for very many values of x.

4. Find the largest prime number generated by Euler's formula.

Goldbach's Conjecture is that every nonzero even number greater than 2 can be written as the sum of two primes. No one has ever proved that this is always true, but no one has found a counterexample, either.

5. Show that Goldbach's Conjecture is true for the first 5 even numbers greater than 2.

6. Give a way that someone could disprove Goldbach's Conjecture.

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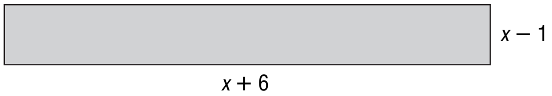
8-4 Enrichment

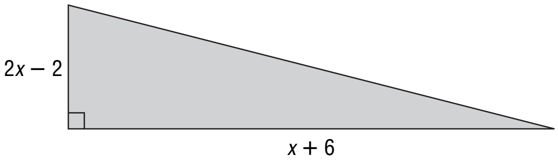
Area Models for Quadratic Trinomials

After you have factored a quadratic trinomial, you can use the factors to draw geometric models of the trinomial.

$x^2 + 5x - 6 = (x - 1)(x + 6)$



To draw a rectangular model, the value 2 was used for x so that the shorter side would have a length of 1. Then the drawing was done in centimeters. So, the area of the rectangle is $x^2 + 5x - 6$.



To draw a right triangle model, recall that the area of a triangle is one-half the base times the height. So, one of the sides must be twice as long as the shorter side of the rectangular model.

$$x^2 + 5x - 6 = (x - 1)(x + 6)$$
$$= \frac{1}{2}(2x - 2)(x + 6)$$

The area of the right triangle is also $x^2 + 5x - 6$.

Factor each trinomial. Then follow the directions to draw each model of the trinomial.

1. $x^2 + 2x - 3$ Use $x = 2$. Draw a rectangle in centimeters.

2. $3x^2 + 5x - 2$ Use $x = 1$. Draw a rectangle in centimeters.

3. $x^2 - 4x + 3$ Use $x = 4$. Draw two different right triangles in centimeters.

4. $9x^2 - 9x + 2$ Use $x = 2$. Draw two different right triangles. Use 0.5 centimeter for each unit.

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8-5 Enrichment

Factoring Trinomials of Fourth Degree

Some trinomials of the form $a^4 + a^2b^2 + b^4$ can be written as the difference of two squares and then factored.

Example 1 **Factor $4x^4 - 37x^2y^2 + 9y^4$.**

Step 1 Find the square roots of the first and last terms.

$\sqrt{4x^4} = 2x^2$ $\qquad \sqrt{9y^4} = 3y^2$

Step 2 Find twice the product of the square roots.

$2(2x^2)(3y^2) = 12x^2y^2$

Step 3 Separate the middle term into two parts. One part is either your answer to Step 2 or its opposite. The other part should be the opposite of a perfect square.

$-37x^2y^2 = -12x^2y^2 - 25x^2y^2$

Step 4 Rewrite the trinomial as the difference of two squares and then factor.

$$\begin{aligned} 4x^4 - 37x^2y^2 + 9y^4 &= (4x^4 - 12x^2y^2 + 9y^4) - 25x^2y^2 \\ &= (2x^2 - 3y^2)^2 - 25x^2y^2 \\ &= [(2x^2 - 3y^2) + 5xy][(2x^2 - 3y^2) - 5xy] \\ &= (2x^2 + 5xy - 3y^2)(2x^2 - 5xy - 3y^2) \end{aligned}$$

Factor each trinomial.

1. $x^4 + x^2y^2 + y^4$

2. $x^4 + x^2 + 1$

3. $9a^4 - 15a^2 + 1$

4. $16a^4 - 17a^2 + 1$

5. $4a^4 - 13a^2 + 1$

6. $9a^4 + 26a^2b^2 + 25b^4$

7. $4x^4 - 21x^2y^2 + 9y^4$

8. $4a^4 - 29a^2c^2 + 25c^4$

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8-6 Enrichment

Squaring Numbers: A Shortcut

A shortcut helps you to square a positive two-digit number ending in 5. The method is developed using the idea that a two-digit number may be expressed as $10t + u$. Suppose $u = 5$.

$$\begin{aligned}(10t + 5)^2 &= (10t + 5)(10t + 5)\\ &= 100t^2 + 50t + 50t + 25\\ &= 100t^2 + 100t + 25\\ (10t + 5)^2 &= 100t(t + 1) + 25\end{aligned}$$

In words, this formula says that the square of a two-digit number ending in 5 has $t(t + 1)$ in the hundreds place. Then 2 is the tens digit and 5 is the units digit.

Example **Using the formula for $(10t + 5)^2$, find 85^2.**

$$\begin{aligned}85^2 &= 100 \cdot 8 \cdot (8 + 1) + 25\\ &= 7200 + 25\\ &= 7225\end{aligned}$$

Shortcut: First think $8 \cdot 9 = 72$. Then write 25.

Thus, to square a number, such as 85, you can write the product of the tens digit and the next consecutive integer $t + 1$. Then write 25.

Find each of the following using the shortcut.

1. 15^2

2. 25^2

3. 35^2

4. 45^2

5. 55^2

6. 65^2

Solve each problem.

7. What is the tens digit in the square of 95?

8. What are the first two digits in the square of 75?

9. Any three-digit number can be written as $100a + 10b + c$. Square this expression to show that if the last digit of a three-digit number is 5 then the last two digits of the square of the number are 2 and 5.

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9-1 Enrichment

Translating Quadratic Graphs

When a figure is moved to a new position without undergoing any rotation, then the figure is said to have been **translated** to that position.

The graph of a quadratic equation in the form $y = (x - b)^2 + c$ is a translation of the graph of $y = x^2$.

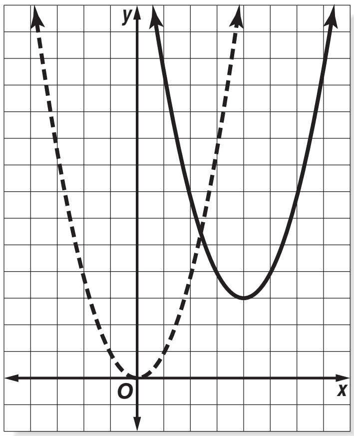
Start with $y = x^2$.

Slide to the right 4 units.

$y = (x - 4)^2$

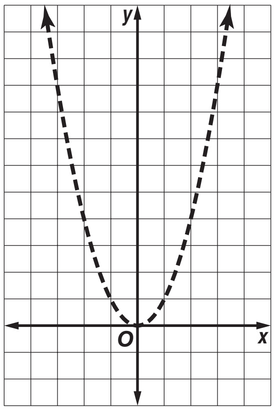
Then slide up 3 units.

$y = (x - 4)^2 + 3$

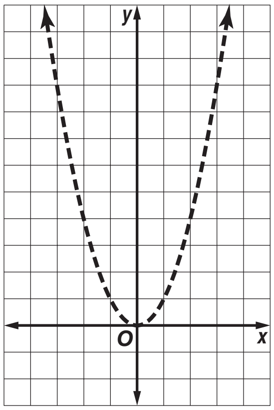


These equations have the form $y = x^2 + c$. Graph each equation.

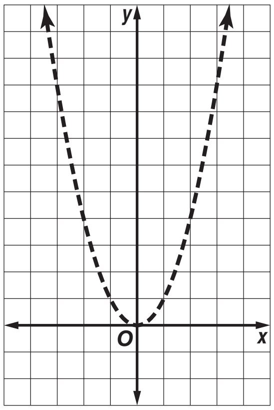
1. $y = x^2 + 1$



2. $y = x^2 + 2$

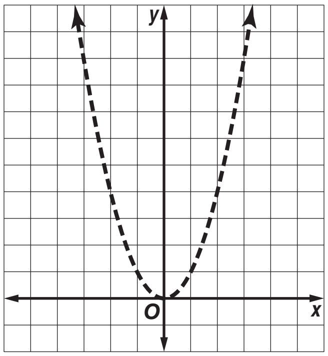


3. $y = x^2 - 2$

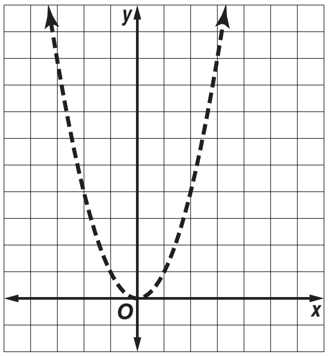


These equations have the form $y = (x - b)^2$. Graph each equation.

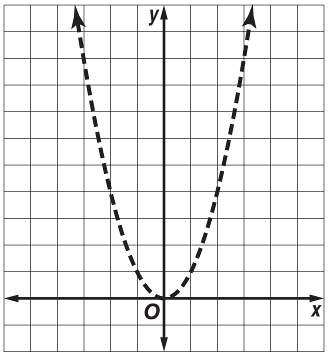
4. $y = (x - 1)^2$



5. $y = (x - 3)^2$



6. $y = (x + 2)^2$



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9-2 Enrichment

Rational Exponents

You have developed the following properties of powers when a is a positive real number and m and n are integers.

$a^m \cdot a^n = a^{m+n}$ $\quad$ $(ab)^m = a^m b^m$ $\quad$ $a^0 = 1$

$(a^m)^n = a^{mn}$ $\quad$ $\dfrac{a^m}{a^n} = a^{m-n}$ $\quad$ $a^{-m} = \dfrac{1}{a^m}$

Exponents need not be restricted to integers. We can define rational exponents so that operations involving them will be governed by the properties for integer exponents.

$\left(a^{\frac{1}{2}}\right)^2 = a^{\frac{1}{2} \cdot 2} = a$ $\quad$ $\left(a^{\frac{1}{3}}\right)^3 = a^{\frac{1}{3} \cdot 3}$ $\quad$ $\left(a^{\frac{1}{n}}\right)^n = a^{\frac{1}{n} \cdot n} = a$

$a^{\frac{1}{2}}$ squared is a. $\quad$ $a^{\frac{1}{3}}$ cubed is a. $\quad$ $a^{\frac{1}{n}}$ to the n power is a.

$a^{\frac{1}{2}}$ is a square root of a. $\quad$ $a^{\frac{1}{3}}$ is a cube root of a. $\quad$ $a^{\frac{1}{n}}$ is an nth root of a.

$a^{\frac{1}{2}} = \sqrt{a}$ $\quad$ $a^{\frac{1}{3}} = \sqrt[3]{a}$ $\quad$ $a^{\frac{1}{n}} = \sqrt[n]{a}$

Now let us investigate the meaning of $a^{\frac{m}{n}}$.

$a^{\frac{m}{n}} = a^{m \cdot \frac{1}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$ $\quad$ $a^{\frac{m}{n}} = a^{\frac{1}{n} \cdot m} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m$

Therefore, $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ or $\left(\sqrt[n]{a}\right)^m$.

Example 1 **Write $\sqrt[4]{a^3}$ in exponential form.**

$\sqrt[4]{a^3} = a^{\frac{3}{4}}$

Example 2 **Write $a^{\frac{2}{5}}$ in radical form.**

$a^{\frac{2}{5}} = \sqrt[5]{a^2}$

Example 3 **Find $\dfrac{a^{\frac{2}{3}}}{a^{\frac{1}{2}}}$.**

$\dfrac{a^{\frac{2}{3}}}{a^{\frac{1}{2}}} = a^{\frac{2}{3} - \frac{1}{2}} = a^{\frac{4}{6} - \frac{3}{6}} = a^{\frac{1}{6}}$ or $\sqrt[6]{a}$

Write each expression in radical form.

1. $b^{\frac{2}{3}}$

2. $3c^{\frac{1}{2}}$

3. $(3c)^{\frac{1}{2}}$

Write each expression in exponential form.

4. $\sqrt[3]{b^4}$

5. $\sqrt{4a^3}$

6. $2 \cdot \sqrt[3]{b^2}$

Perform the operation indicated. Answers should show positive exponents only.

7. $\left(a^3 b^{\frac{1}{4}}\right)^2$

8. $\dfrac{-8a^{\frac{3}{4}}}{2a^{\frac{1}{2}}}$

9. $\left(\dfrac{b^{\frac{1}{2}}}{b^{-\frac{2}{3}}}\right)^3$

10. $\sqrt{a^3} \cdot \sqrt{a}$

11. $\left(a^2 b^{-\frac{1}{3}}\right)^{-\frac{1}{2}}$

12. $-2a^{\frac{1}{3}}b^0\left(5a^{\frac{1}{2}}b^{-\frac{2}{3}}\right)$

9-3 Enrichment

Parabolas Through Three Given Points

If you know two points on a straight line, you can find the equation of the line. To find the equation of a parabola, you need three points on the curve.

Here is how to approximate an equation of the parabola through the points (0, −2), (3, 0), and (5, 2).

Use the general equation $y = ax^2 + bx + c$. By substituting the given values for x and y, you get three equations.

(0, −2): $-2 = c$

(3, 0): $0 = 9a + 3b + c$

(5, 2): $2 = 25a + 5b + c$

First, substitute −2 for c in the second and third equations. Then solve those two equations as you would any system of two equations. Multiply the second equation by 5 and the third equation by −3.

$$\begin{array}{lcr} 0 = 9a + 3b - 2 & \xrightarrow{\text{Multiply by 5.}} & 0 = 45a + 15b - 10 \\ 0 = 25a + 5b - 2 & \xrightarrow{\text{Multiply by } -3.} & \underline{-6 = -75a - 15b + 6} \\ & & -6 = -30a \qquad - 4 \\ & & a = \frac{1}{15} \end{array}$$

To find b, substitute $\frac{1}{15}$ for a in either the second or third equation.

$$0 = 9\left(\frac{1}{15}\right) + 3b - 2$$

$$b = \frac{7}{15}$$

The equation of a parabola through the three points is

$$y = \frac{1}{15}x^2 + \frac{7}{15}x - 2.$$

Find the equation of a parabola through each set of three points.

1. (1, 5), (0, 6), (2, 3)

2. (−5, 0), (0, 0), (8, 100)

3. (4, −4), (0, 1), (3, −2)

4. (1, 3), (6, 0), (0, 0)

5. (2, 2), (5, −3), (0, −1)

6. (0, 4), (4, 0), (−4, 4)

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9-4 Enrichment

Golden Rectangles

A **golden rectangle** has the property that its sides satisfy the following proportion.

$$\frac{a+b}{a} = \frac{a}{b}$$

Two quadratic equations can be written from the proportion. These are sometimes called **golden quadratic** equations.

1. In the proportion, let $a = 1$. Use cross-multiplication to write a quadratic equation.

2. Solve the equation in Exercise 1 for b.

3. In the proportion, let $b = 1$. Write a quadratic equation in a.

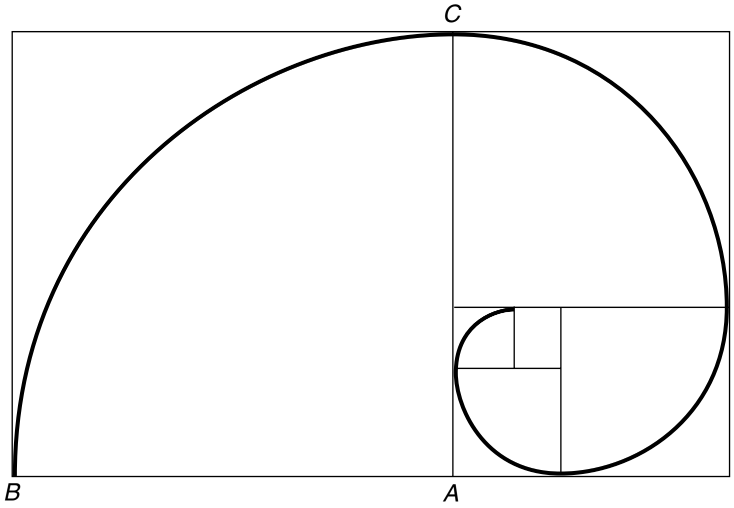
4. Solve the equation in Exercise 3 for a.

5. Explain why $\frac{1}{2}(\sqrt{5} + 1)$ and $\frac{1}{2}(\sqrt{5} - 1)$ are called golden ratios.

Another property of golden rectangles is that a square drawn inside a golden rectangle creates another, smaller golden rectangle.

In the design at the right, opposite vertices of each square have been connected with quarters of circles.

For example, the arc from point B to point C is created by putting the point of a compass at point A. The radius of the arc is the length BA.



6. On a separate sheet of paper, draw a larger version of the design. Start with a golden rectangle with a long side of 10 inches.

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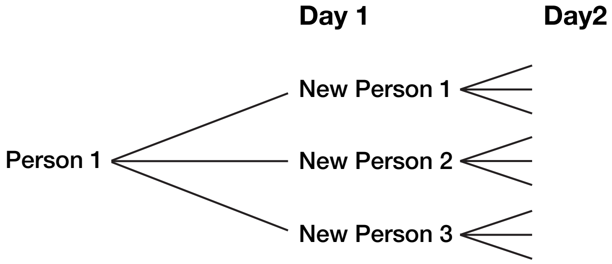
9-5 Enrichment

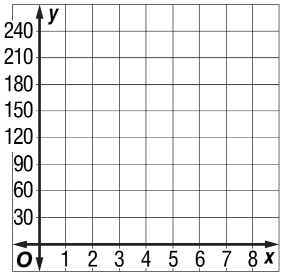
Pay it Forward

The idea behind “pay it forward” is that on the first day, one person does a good deed for three different people. Then, on the second day, those three people will each perform good deeds for 3 more people, so that on Day 2, there are 3 × 3 or 9 good deeds being done. Continue this process to fill in the chart. A tree diagram will help you fill in the chart.

Day	# of Deeds
0	1
1	3
2	9
3	
4	
5	



1. Graph the data you found in the chart as ordered pairs and connect with a smooth curve.



2. What type of function is your graph from problem 1? Write an equation that can be used to determine the number of good deeds on any given day, x.

3. How many good deeds will be performed on Day 21?

The formula, $\frac{3^{n+1}-3}{2}$, can be used to determine the **total** number of good deeds that have been performed, where n represents the day. For example, on Day 2, there have been 3 + 9 or 12 good deeds performed. Using the formula, you get, $\frac{3^{2+1}-3}{2}$ or $\frac{3^3-3}{2}$ + $\frac{27-3}{2}$ or a total of 12 good deeds performed.

4. Use this formula to determine the approximate number of good deeds that have been performed through Day 21.

5. Look up the world population. How does your number from Exercise 4 compare to the world population?

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9-6 Enrichment

Growth and Decay

Sierpinski Triangle is an example of a fractal that changes exponentially. Start with an equilateral triangle and find the midpoints of each side. Then connect the midpoints to form a smaller triangle. Remove this smaller triangle from the larger one.

Repeat the process to create the next triangle in the sequence. Find the midpoints of the sides of the three remaining triangles and connect them to form smaller triangles to be removed.

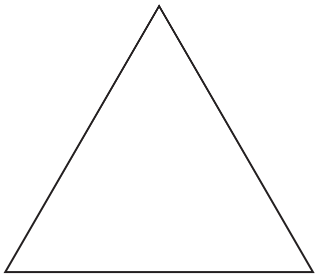

figure 1

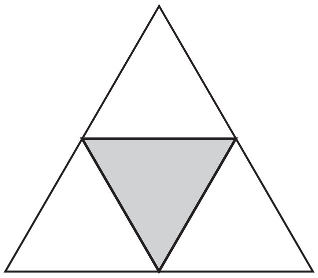

figure 2

Cut out: $\frac{1}{4}$

Area = $1 - \frac{1}{4}$ or $\frac{3}{4}$

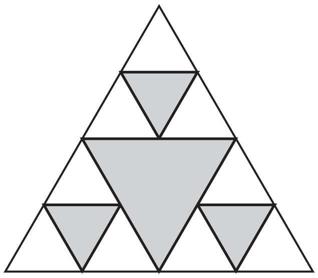

figure 3

Cut out: $\frac{1}{4} + \frac{3}{16}$ or $\frac{7}{16}$

Area = $1 - \frac{7}{16}$ or $\frac{9}{16}$

1. Find the next triangle in the sequence. How much has been cut out? What is the area of the fourth figure in the sequence?

2. Make a conjecture as to what you need to multiply the previous amount cut by to find the new amount cut.

3. Fill in the chart to represent the amount cut and the area remaining for each triangle in the sequence.

figure	1	2	3	4	5	6
amount cut	0	$\frac{1}{4}$	$\frac{7}{16}$			
area remaining	1	$\frac{3}{4}$	$\frac{9}{16}$			

4. Write an equation to represent the area that is left in the nth triangle in the sequence.

5. If this process is continued, make a conjecture as to the remaining area.

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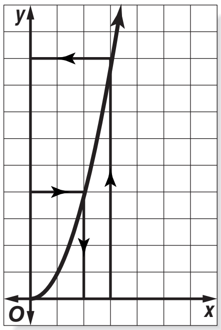
10-1 Enrichment

Squares and Square Roots From a Graph



The graph of $y = x^2$ can be used to find the squares and square roots of numbers.

To find the square of 3, locate 3 on the x-axis. Then find its corresponding value on the y-axis.

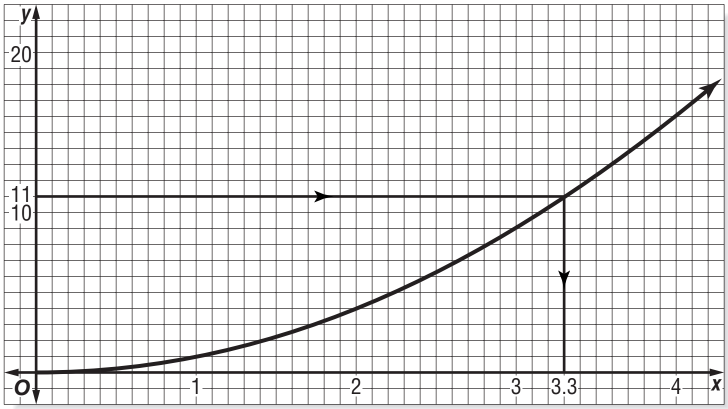
The arrows show that $3^2 = 9$.

To find the square root of 4, first locate 4 on the y-axis. Then find its corresponding value on the x-axis. Following the arrows on the graph, you can see that $\sqrt{4} = 2$.

A small part of the graph at $y = x^2$ is shown below. A 1:10 ratio for unit length on the y-axis to unit length on the x-axis is used.

Example **Find $\sqrt{11}$.**

The arrows show that $\sqrt{11} = 3.3$ to the nearest tenth.



Exercises

Use the graph above to find each of the following to the nearest whole number.

1. 1.5^2

2. 2.7^2

3. 0.9^2

4. 3.6^2

5. 4.2^2

6. 3.9^2

Use the graph above to find each of the following to the nearest tenth.

7. $\sqrt{15}$

8. $\sqrt{8}$

9. $\sqrt{3}$

10. $\sqrt{5}$

11. $\sqrt{14}$

12. $\sqrt{17}$

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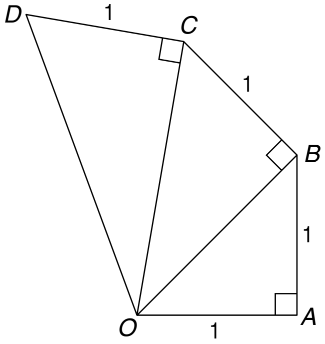
10-2 Enrichment

The Wheel of Theodorus

The Greek mathematicians were intrigued by problems of representing different numbers and expressions using geometric constructions.

Theodorus, a Greek philosopher who lived about 425 B.C., is said to have discovered a way to construct the sequence $\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots$.

The beginning of his construction is shown. You start with an isosceles right triangle with sides 1 unit long.



Use the figure above. Write each length as a radical expression in simplest form.

1. line segment AO

2. line segment BO

3. line segment CO

4. line segment DO

5. Describe how each new triangle is added to the figure.

6. The length of the hypotenuse of the first triangle is $\sqrt{2}$. For the second triangle, the length is $\sqrt{3}$. Write an expression for the length of the hypotenuse of the nth triangle.

7. Show that the method of construction will always produce the next number in the sequence. (*Hint:* Find an expression for the hypotenuse of the $(n + 1)$th triangle.)

8. In the space below, construct a Wheel of Theodorus. Start with a line segment 1 centimeter long. When does the Wheel start to overlap?

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10-3 Enrichment

More Than One Square Root

You have learned that to remove the square root in an equation, you first need to isolate the square root, then square both sides of the equation, and finally, solve the resulting equation. However, there are equations that contain more that one square root and simply squaring once is not enough to remove all of the radicals.

Example **Solve $\sqrt{x+7} = \sqrt{x} + 1$.**

$\sqrt{x+7} = \sqrt{x} + 1$	One of the square roots is already isolated.
$(\sqrt{x+7})^2 = (\sqrt{x} + 1)^2$	Square both sides to remove the square root.
$x + 7 = x + 2\sqrt{x} + 1$	Simplify. Use the FOIL method to square right side.
$x + 7 - x - 1 = 2\sqrt{x}$	Simplify.
$6 = 2\sqrt{x}$	Simplify. Isolate the square root term again.
$3 = \sqrt{x}$	Divide both sides by 2.
$9 = x$	Square both sides to remove the square root.

Check: Substitute into the *original* equation to make sure your solution is valid.

$\sqrt{9+7} = \sqrt{9} + 1$	Replace x with 9.
$\sqrt{16} = 3 + 1$	Simpify.
$4 = 4$✔	The equation is true, so $x = 9$ is the solution.

Exercises **Solve each equation.**

1. $\sqrt{x+13} - 2 = \sqrt{x+1}$

2. $\sqrt{x+11} = \sqrt{x+3} + 2$

3. $\sqrt{x+9} - 3 = \sqrt{x-6}$

4. $\sqrt{x+21} = \sqrt{x} + 3$

5. $\sqrt{x+9} + 3 = \sqrt{x} + 4$

6. $\sqrt{x-6} + 6 = \sqrt{x+1} + 5$

Lesson 10-3

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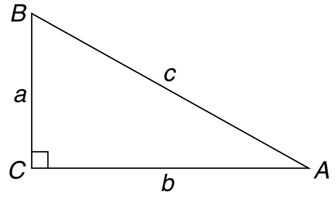
10-4 Enrichment

Pythagorean Triples

Recall the Pythagorean Theorem:

In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.



$a^2 + b^2 = c^2$

Note that c is the length of the hypotenuse.

The integers 3, 4, and 5 satisfy the Pythagorean Theorem and can be the lengths of the sides of a right triangle.

$$3^2 + 4^2 = 5^2$$
$$9 + 16 = 25$$
$$25 = 25$$

Furthermore, for any positive integer n, the numbers $3n$, $4n$, and $5n$ satisfy the Pythagorean Theorem.

For $n = 2$: $6^2 + 8^2 = 10^2$
$$36 + 64 = 100$$
$$100 = 100$$

If three numbers satisfy the Pythagorean Theorem, they are called a **Pythagorean triple**. Here is an easy way to find other Pythagorean triples.

The numbers a, b, and c are a Pythagorean triple if $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$, where m and n are relatively prime positive integers and $m > n$.

Example **Choose $m = 5$ and $n = 2$.**

$$a = m^2 - n^2 = 5^2 - 2^2 = 25 - 4 = 21$$
$$b = 2mn = 2(5)(2) = 20$$
$$c = m^2 + n^2 = 5^2 + 2^2 = 25 + 4 = 29$$

Check $20^2 + 21^2 \stackrel{?}{=} 29^2$
$$400 + 441 \stackrel{?}{=} 841$$
$$841 = 841$$

Exercises

Use the following values of *m* and *n* to find Pythagorean triples.

1. $m = 3$ and $n = 2$

2. $m = 4$ and $n = 1$

3. $m = 5$ and $n = 3$

4. $m = 6$ and $n = 5$

5. $m = 10$ and $n = 7$

6. $m = 8$ and $n = 5$

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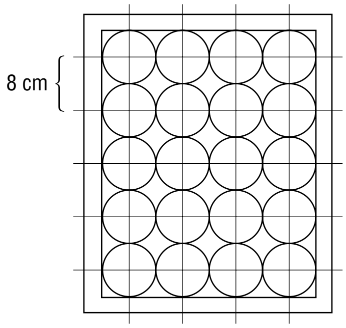
10-5 Enrichment

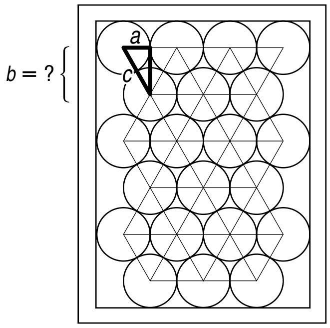
A Space-Saving Method

Two arrangements for cookies on a 32 cm by 40 cm cookie sheet are shown at the right. The cookies have 8-cm diameters after they are baked. The centers of the cookies are on the vertices of squares in the top arrangement. In the other, the centers are on the vertices of equilateral triangles. Which arrangement is more economical? The triangle arrangement is more economical, because it contains one more cookie.



20 cookies

In the square arrangement, rows are placed every 8 cm. At what intervals are rows placed in the triangle arrangement?



21 cookies

Look at the right triangle labeled a, b, and c. A leg a of the triangle is the radius of a cookie, or 4 cm. The hypotenuse c is the sum of two radii, or 8 cm. Use the Pythagorean theorem to find b, the interval of the rows.

$$\begin{aligned}
c^2 &= a^2 + b^2 \\
8^2 &= 4^2 + b^2 \\
64 - 16 &= b^2 \\
\sqrt{48} &= b \\
4\sqrt{3} &= b \\
b &= 4\sqrt{3} \approx 6.93
\end{aligned}$$

The rows are placed approximately every 6.93 cm.

Solve each problem.

1. Suppose cookies with 10-cm diameters are arranged in the triangular pattern shown above. What is the interval b of the rows?

2. Find the diameter of a cookie if the rows are placed in the triangular pattern every $3\sqrt{3}$ cm.

3. Describe other practical applications in which this kind of triangular pattern can be used to economize on space.

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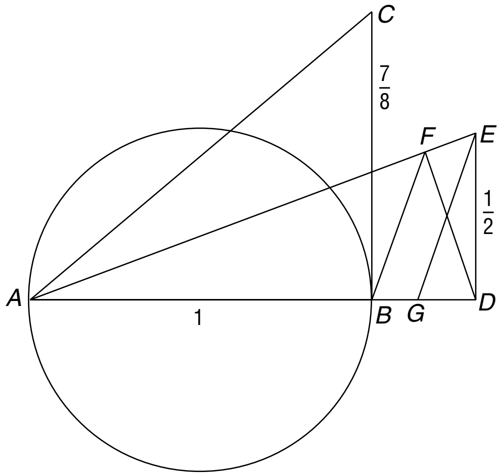
10-6 Enrichment

A Curious Construction

Many mathematicians have been interested in ways to construct the number π. Here is one such geometric construction.

In the drawing, triangles ABC and ADE are right triangles. The length of $\overline{AD}$ equals the length of $\overline{AC}$ and $\overline{FB}$ is parallel to $\overline{EG}$.

The length of $\overline{BG}$ gives a decimal approximation of the fractional part of π to six decimal places.



Follow the steps to find the length of $\overline{BG}$. Round to seven decimal places.

1. Use the length of $\overline{BC}$ and the Pythagorean Theorem to find the length of $\overline{AC}$.

2. Find the length of $\overline{AD}$.

3. Use the length of $\overline{AD}$ and the Pythagorean Theorem to find the length of $\overline{AE}$.

4. The sides of the similar triangles FED and DEA are in proportion. So, $\frac{FE}{0.5} = \frac{0.5}{AE}$. Find the length of $\overline{FE}$.

5. Find the length of $\overline{AF}$.

6. The sides of the similar triangles AFB and AEG are in proportion. So, $\frac{AF}{AE} = \frac{AB}{AG}$. Find the length of $\overline{AG}$.

7. Now, find the length of $\overline{BG}$.

8. The value of π to seven decimal places is 3.1415927. Compare the fractional part of π with the length of $\overline{BG}$.

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11-1 Enrichment

Direct or Indirect Variation

Fill in each table below. Then write *inversely*, or *directly* to complete each conclusion.

1.

l	2	4	8	16	32
w	4	4	4	4	4
A					

For a set of rectangles with a width of 4, the area varies ______________ as the length.

2.

Hours	2	4	5	6
Speed	55	55	55	55
Distance				

For a car traveling at 55 mi/h, the distance covered varies ______________ as the hours driven.

3.

Oat Bran	$\frac{1}{3}$ cup	$\frac{2}{3}$ cup	1 cup
Water	1 cup	2 cups	3 cups
Servings	1	2	

The number of servings of oat bran varies ______________ as the number of cups of oat bran.

4.

Hours of Work	128	128	128
People Working	2	4	8
Hours per Person			

A job requires 128 hours of work. The number of hours each person works varies ______________ as the number of people working.

5.

Miles	100	100	100	100
Rate	20	25	50	100
Hours	5			

For a 100-mile car trip, the time the trip takes varies ______________ as the average rate of speed the car travels.

6.

b	3	4	5	6
h	10	10	10	10
A	15			

For a set of right triangles with a height of 10, the area varies ______________ as the base.

Use the table at the right.

x	1	1.5	2	2.5	3
y	2	3	4	5	6
z	60	40	30	24	20

7. x varies ______________ as y.

8. z varies ______________ as y.

9. x varies ______________ as z.

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11-2 Enrichment

Shannon's Juggling Theorem

Mathematicians look at various mathematical ways to represent juggling. One way they have found to represent juggling is Shannon's Juggling Theorem. Shannon's Juggling Theorem uses the rational equation

$$\frac{f+d}{v+d} = \frac{b}{h}$$

where f is the flight time, or how long a ball is in the air, d is the dwell time, or how long a ball is in a hand, v is the vacant time, or how long a hand is empty, b is the number of balls, and h is the number of hands (either 1 or 2 for a real-life situation, possibly more for a computer simulation).

So, given the values for f, d, v, and h, it is possible to determine the number of balls being juggled. If the flight time is 9 seconds, the dwell time is 3 seconds, the vacant time is 1 second, and the number of hands is 2, how many balls are being juggled?

$$\frac{f+d}{v+d} = \frac{b}{h}$$ Original equation

$$\frac{9+3}{1+3} = \frac{b}{2}$$ Replace with the values given.

$$\frac{12}{4} = \frac{b}{2}$$ Simplify.

$$24 = 4b$$ Cross multiply.

$$6 = b$$ Divide.

So, the number of balls being juggled is 6.

Given the following information, determine the number of balls being juggled.

1. flight time = 6 seconds, vacant time = 1 second, dwell time = 1 second, number of hands = 2

2. flight time = 13 seconds, vacant time = 1 second, dwell time = 5 seconds, number of hands = 1

3. flight time = 4 seconds, vacant time = 1 second, dwell time = 1 second, number of hands = 2

4. flight time = 16 seconds, vacant time = 1 second, dwell time = 2 seconds, number of hands = 2

5. flight time = 18 seconds, vacant time = 3 seconds, dwell time = 2 seconds, number of hands = 1

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11-3 Enrichment

Continued Fractions

The following is an example of a continued fraction. By starting at the bottom you can simplify the expression to a rational number.

$$3 + \cfrac{4}{1 + \frac{6}{7}} = 3 + \cfrac{4}{\frac{13}{7}}$$

$$= 3 + \frac{28}{13} \text{ or } \frac{67}{13}$$

Example **Express $\frac{48}{19}$ as a continued fraction.**

$$\frac{48}{19} = 2 + \frac{10}{19}$$

Notice that the numerator of the last fraction must be equal to 1 before the process stops.

$$= 2 + \cfrac{1}{\frac{19}{10}}$$

$$= 2 + \cfrac{1}{1 + \frac{9}{10}}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{\frac{10}{9}}}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{9}}}$$

Exercises

Write each continued fraction as a rational number.

1. $6 + \cfrac{1}{1 + \cfrac{1}{3 + \frac{1}{3}}}$

2. $5 + \cfrac{7}{2 + \cfrac{3}{4 + \frac{2}{3}}}$

Write each rational number as a continued fraction.

3. $\frac{97}{17}$

4. $\frac{22}{64}$

Lesson 11-3

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11-4 Enrichment

Geometric Series

A *geometric series* is a sum of the terms in a *geometric sequence*. Each term of a geometric sequence is formed by multiplying the previous term by a constant term called the *common ratio*.

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \leftarrow$ geometric sequence where the common ratio is $\frac{1}{2}$

The sum of a geometric series can be represented by the rational expression

$x_0\frac{(r)^n - 1}{r - 1}$, where x_0 is the first term of the series, r is the common ratio, and n is the number of terms.

In the example above, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1^* \frac{\left(\frac{1}{2}\right)^4 - 1}{\frac{1}{2} - 1}$ or $\frac{15}{8}$.

You can check this by entering $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ a calculator. The result is the same.

Rewrite each sum as a rational expression and simplify.

1. $9 + 3 + 1 + \frac{1}{3} + \frac{1}{9}$

2. $500 + 250 + 125 + 62\frac{1}{2}$

3. $6 + 1 + \frac{1}{6} + \frac{1}{36}$

4. $100 + 20 + 4 + \frac{4}{5}$

5. $1000 + 100 + 10 + 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000}$

6. $55 + 5 + \frac{5}{11} + \frac{5}{121}$

11-5 Enrichment

Synthetic Division

You can divide a polynomial such as $3x^3 - 4x^2 - 3x - 2$ by a binomial such as $x - 3$ by a process called **synthetic division**. Compare the process with long division in the following explanation.

Example **Divide $(3x^3 - 4x^2 - 3x - 2)$ by $(x - 3)$ using synthetic division.**

$$\begin{array}{r|rrrr} & 3 & -4 & -3 & -2 \\ & & 9 & 15 & 36 \\ \hline 3 & 3 & 5 & 12 & 34 \end{array}$$

$$\underbrace{\qquad\qquad\qquad}_{3x^2 + 5x + 12,\ \text{remainder } 34}$$

1. Show the coefficients of the terms in descending order.
2. The divisor is $x - 3$. Since 3 is to be subtracted, write 3 in the corner $\lrcorner$.
3. Bring down the first coefficient, 3.
4. Multiply. $3 \cdot 3 = 9$
5. Add. $-4 + 9 = 5$
6. Multiply. $3 \cdot 5 = 15$
7. Add. $-3 + 15 = 12$
8. Multiply. $3 \cdot 12 = 36$
9. Add. $-2 + 36 = 34$

Check Use long division.

$$\begin{array}{rl} & 3x^2 + \ \ 5x + 12 \\ x - 3 \! & \!\overline{)\,3x^3 - 4x^2 - \ \ 3x - \ \ 2} \\ & \underline{3x^3 - 9x^2} \\ & \qquad\quad 5x^2 - \ \ 3x \\ & \qquad\quad \underline{5x^2 - 15x} \\ & \qquad\qquad\quad 12x - \ \ 2 \\ & \qquad\qquad\quad \underline{12x - 36} \\ & \qquad\qquad\qquad\quad\ 34 \end{array}$$

The result is $3x^2 + 5x + 12 + \dfrac{34}{x - 3}$.

Divide by using synthetic division. Check your result using long division.

1. $(x^3 + 6x^2 + 3x + 1) \div (x - 2)$

2. $(x^3 - 3x^2 - 6x - 20) \div (x - 5)$

3. $(2x^3 - 5x + 1) \div (x + 1)$

4. $(3x^3 - 7x^2 + 4) \div (x - 2)$

5. $(x^3 + 2x^2 - x + 4) \div (x + 3)$

6. $(x^3 + 4x^2 - 3x - 11) \div (x - 4)$

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11-6 Enrichment

Sum and Difference of Any Two Like Powers

The sum of any two like powers can be written $a^n + b^n$, where n is a positive integer. The difference of like powers is $a^n - b^n$. Under what conditions are these expressions exactly divisible by $(a + b)$ or $(a - b)$? The answer depends on whether n is an odd or even number.

Use long division to find the following quotients. (*Hint:* Write $a^3 + b^3$ as $a^3 + 0a^2 + 0a + b^3$.) Is the numerator exactly divisible by the denominator? Write *yes* or *no*.

1. $\frac{a^3 + b^3}{a + b}$ **2.** $\frac{a^3 + b^3}{a - b}$ **3.** $\frac{a^3 - b^3}{a + b}$ **4.** $\frac{a^3 - b^3}{a - b}$

5. $\frac{a^4 + b^4}{a + b}$ **6.** $\frac{a^4 + b^4}{a - b}$ **7.** $\frac{a^4 - b^4}{a + b}$ **8.** $\frac{a^4 - b^4}{a - b}$

9. $\frac{a^5 + b^5}{a + b}$ **10.** $\frac{a^5 + b^5}{a - b}$ **11.** $\frac{a^5 - b^5}{a + b}$ **12.** $\frac{a^5 - b^5}{a - b}$

13. Use the words *odd* and *even* to complete these two statements.

a. $a^n + b^n$ is divisible by $a + b$ if n is ___________, and by neither $a + b$ nor $a - b$ if n is ___________.

b. $a^n - b^n$ is divisible by $a - b$ if n is ___________, and by both $a + b$ and $a - b$ if n is ___________.

14. Describe the signs of the terms of the quotients when the divisor is $a - b$.

15. Describe the signs of the terms of the quotient when the divisor is $a + b$.

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11-7 Enrichment

Partial Fractions

The method of partial fractions is a way of rewriting a rational expression as a sum of rational expressions. Look at the following example.

Rewrite $\frac{11x + 4}{x^2 + x - 2}$ as the sum of two rational expressions.

$\frac{11x + 4}{x^2 + x - 2} = \frac{11x + 4}{(x - 1)(x + 2)}$ Factor the denominator.

So, $\frac{A}{x - 1} + \frac{B}{x + 2} = \frac{11x + 4}{(x - 1)(x + 2)}$ Now, solve for A and B.

$A(x + 2) + B(x - 1) = 11x + 4$ Multiply by the LCD, $(x - 1)(x + 2)$

$Ax + 2A + Bx - B = 11x + 4$ Distribute.

$Ax + Bx + 2A - B = 11x + 4$ Commutative Property of Addition.

$(A + B)x + (2A - B) = 11x + 4$ Distributive Property.

So, $A + B = 11$

and $2A - B = 4$. Write the corresponding system of equations.

Now solve the system of equations.

$$\begin{array}{rl} A + B &= 11 \\ \underline{2A - B} &\underline{= \ \ 4} \\ 3A &= 15 \end{array}$$

So, $A = 5$ and $B = 6$.

So, $\frac{11x + 4}{x^2 + x - 2} = \frac{5}{x - 1} + \frac{6}{x + 2}$.

Rewrite as partial fractions.

1. $\frac{5x - 4}{x^2 - x - 2}$

2. $\frac{4x - 14}{x^2 - 2x - 24}$

3. $\frac{13x - 5}{x^2 - 25}$

4. $\frac{9x + 1}{x^2 - 2x - 3}$

5. $\frac{21x + 35}{x^2 - 49}$

6. $\frac{7x + 6}{x^2 + 3x}$

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11-8 Enrichment

Complex Fractions

Complex fractions are really not complicated. Remember that a fraction can be interpreted as dividing the numerator by the denominator.

$$\frac{\frac{2}{3}}{\frac{5}{7}} = \frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{5} = \frac{2(7)}{3(5)} = \frac{14}{15}$$

Let a, b, c, and d be numbers, with $b \neq 0$, $c \neq 0$, and $d \neq 0$.

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

Notice the pattern: numerator of the answer (*ad*) $\frac{\frac{a}{b}}{\frac{c}{d}}$ denominator of the answer (*bc*)

Example 1 **Simplify** $\frac{\frac{5x}{4}}{\frac{x+2}{3}}$.

$$\frac{\frac{5x}{4}}{\frac{x+2}{3}} = \frac{5x(3)}{4(x+2)}$$

$$= \frac{15x}{4x+8}$$

Example 2 **Simplify** $\frac{\frac{x}{2}+4}{3x-2}$.

$$\frac{\frac{x}{2}+4}{3x-2} = \frac{\frac{x+8}{2}}{\frac{3x-2}{1}}$$

$$= \frac{(x+8)(1)}{2(3x-2)} = \frac{x+8}{6x-4}$$

Simplify each complex fraction.

1. $\frac{\frac{2x}{5}}{\frac{y}{6}}$

2. $\frac{\frac{4}{5x}}{\frac{3}{x}}$

3. $\frac{x-3}{\frac{2x+1}{4}}$

4. $\frac{x^2+\frac{1}{3}}{4x+\frac{1}{3}}$

5. $\frac{1-x^{-1}}{\frac{2x}{5}-1}$

6. $\frac{x+2x^{-2}}{2+\frac{x}{3}}$

7. $\frac{x}{x+\frac{1}{x+\frac{1}{x}}}$

8. $\frac{x+2}{x-2+\frac{1}{x+2+\frac{1}{x}}}$

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11-9 Enrichment

Winning Distances

In 1999, Hicham El Guerrouj set a world record for the mile run with a time of 3:43.13 (3 min 43.13 s). In 1954, Roger Bannister ran the first mile under 4 minutes at 3:59.4. Had they run those times in the same race, how far in front of Bannister would El Guerrouj have been at the finish?

Use $\frac{d}{t}$ = r. Since 3 min 43.13 s = 223.13 s, and 3 min 59.4 s = 239.4 s,

El Guerrouj's rate was $\frac{5280 \text{ ft}}{223.13 \text{ s}}$ and Bannister's rate was $\frac{5280 \text{ ft}}{239.4 \text{ s}}$.

	r	*t*	*d*
El Guerrouj	$\frac{5280}{223.13}$	223.13	5280 feet
Bannister	$\frac{5280}{239.4}$	223.13	$\frac{5280}{239.4} \cdot 223.13$ or 4921.2 feet

Therefore, when El Guerrouj hit the tape, he would be 5280 − 4921.2, or 358.8 feet, ahead of Bannister. Let's see whether we can develop a formula for this type of problem.

Let D = the distance raced,
W = the winner's time,
and L = the loser's time.

Following the same pattern, you obtain the results shown in the table at the right.

	r	*t*	*d*
Winner	$\frac{D}{W}$	W	$\frac{D}{W} \cdot W = D$
Loser	$\frac{D}{L}$	W	$\frac{D}{L} \cdot W = \frac{DW}{L}$

The winning distance will be $D - \frac{DW}{L}$.

1. Show that the expression for the winning distance is equivalent to $\frac{D(L - W)}{L}$.

Use the formula winning distance = $\frac{D(L - W)}{L}$ to find the winning distance to the nearest tenth for each of the following Olympic races.

2. women's 400 meter relay: Canada 48.4 s (1928);
East Germany 41.6 s (1980)

3. men's 200 meter freestyle swimming: Yevgeny Sadovyi 1 min 46.70 s (1992);
Michael Gross 1 min 47.44 s (1984)

4. men's 50,000 meter walk: Vyacheslav Ivanenko 3 h 38 min 29 s (1988);
Hartwig Gauter 3 h 49 min 24 s (1980)

5. women's 400 meter freestyle relay: United States 3 min 39.29 s (1996);
East Germany 3 min 42.71 s (1980)

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12-1 Enrichment

Heads or Tails

Based on the way certain coins are manufactured, there is a ‘bias’ towards landing either heads up or tails up when balanced on their edges. Balance several pennies from the same minting year on their edges on a table. Gently shake the table so that the pennies fall flat.

1. What percent of the pennies would you expect to land heads up?

2. What percent actually landed heads up?

Repeat the experiment.

3. Were the results similar to your previous attempt?

4. Make a conjecture as to why the percent of pennies that land face up differs from what you would expect it to be.

When a penny is manufactured, it is beveled or grooved slightly, so that when it falls, it is more likely to fall heads up. There appears to be a difference between minting years. Repeat this experiment with other minting years.

5. Are the results about the same as before? Explain why or why not.

Because of this bias, tossing a coin and letting it hit the ground or a table to determine a winner is not fair. The penny is most likely to land with the head side up.

6. How can you change a coin toss, so that heads and tails are more equally likely to appear?

Try this experiment with other coins to determine if there is a “bias” in manufacturing these coins as well.

7. Explain what happens when you try this experiment with a nickel, a dime, and a quarter.

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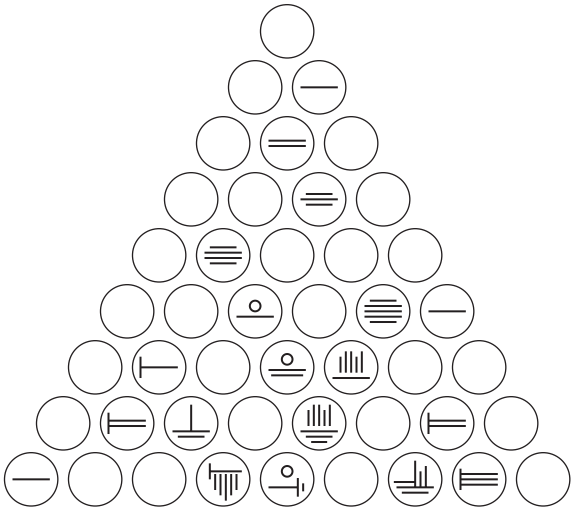
12-2 Enrichment

Pascal's Triangle

Pascal's Triangle is a pattern of numbers used at many levels of mathematics. It is named for Blaise Pascal, a seventeenth-century French mathematician who discovered several applications of the pattern. However, records of the triangle have been traced as far back as twelfth-century China and Persia. In the year 1303, the Chinese mathematician Zhū-Shìjié wrote *The Precious Mirror of the Four Elements,* in which he described how the triangle could be used to solve polynomial equations. The figure at the right is adapted from the original Chinese manuscript. In the figure, some circles are empty while others contain Chinese symbols.



At the right, a portion of Pascal's Triangle is shown using Hindu-Arabic numerals.

The triangle expresses a relationship between numbers that you can discover by comparing the Chinese version and the Hindu-Arabic version.

1
1 1
1 _____ 1
1 _____ _____ 1
1 4 _____ _____ 1
1 _____ _____ 10 _____ 1

1. What Chinese symbol corresponds to the Hindu-Arabic numeral 1?

2. Fill in the outermost circles in the Chinese version of Pascal's Triangle.

3. What Chinese symbol corresponds to the Hindu-Arabic numeral 4?

4. What Chinese symbol corresponds to the Hindu-Arabic numeral 10?

5. Based upon your investigation so far, fill in as many of the missing numbers as you can in both the Chinese and Hindu-Arabic versions of Pascal's Triangle.

6. Pascal's Triangle is *symmetric* about an imaginary vertical line that separates the left and right halves of the triangle. Use this fact to fill in more missing numbers in the triangles.

7. Each row of the triangle is generated from the row above by using a simple rule. Find the rule. Then fill in the remaining entries in both triangles.

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12-3 Enrichment

A Great Pizza Deal

A television commercial advertised a pizza deal in which customers could choose two pizzas each with up to five toppings chosen from a set of eleven toppings. On the commercial, a boy claims that there are 1024^2 or 1,048,576 ways for the customer to choose the two pizzas. Is this a valid claim?

Pizza Deal!!

For a limited time, you can choose 2 pizzas with up to 5 toppings for only

$7.99!!!!!!!

Topping	**Yes**	**No**
Pepperoni	☐	☐
Sausage	☐	☐
Tomato	☐	☐
Green olives	☐	☐
Black olives	☐	☐
Mushrooms	☐	☐
Extra Cheese	☐	☐
Peppers	☐	☐
Onions	☐	☐
Pineapple	☐	☐
Ham	☐	☐

1. Suppose the customer can have as many toppings, up to eleven, as desired. Using counting techniques, how many different ways can the pizzas be made? (Hint: Imagine that the customer has to choose “yes” or “no” for each topping.)

2. Use combinations to determine how many ways the customer can make one pizza with up to five toppings.

3. How many ways can the two pizzas be built if the they are identical?

4. How many ways can the two pizzas be built if the they are different? (Hint: Use the total number of ways that one pizza can be made from problem 2 and then choose 2 of those possibilities.) Should you use a combination or a permutation?

5. Add the results from problems 3 and 4 to determine the total number of ways to make two pizzas with up to five toppings each.

6. The boy in the ad claims that the customer has 1024^2 choices. What is 1024^2? How does this compare to the claim made by the boy in the advertisement?

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12-4 Enrichment

Happy Birthday

On a reality show, a contestant must randomly choose enough people to go into a room so that the probability that at least one of those people has the same birthday (month and date) as the contestant is greater than 50%. Time is of the essence, so the contestant doesn't want any more people than necessary. What number of people should the contestant gather?

1. Make a conjecture about the number of people needed.

2. When the probability that every person in the room has a different birthday is less than 50%, the probability that two of them will share a birthday is greater than 50%. Explain why.

3. Observe this pattern:
If there are two people in the room, the probability that they all have different birthdays is $\frac{365}{365} \cdot \frac{364}{365}$ or 99.7%.

If there are three people in the room, the probability that they all have different birthdays is $\frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365}$ or 99.2%.

If there are four people in the room, the probability that they all have different birthdays is $\frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365}$ or 99.2%.

Extend this pattern until you get a probability less than 50%. How many people do you need to get a probability less than 50%?

4. So, how many people do you need to have in a room so that the probability of two of them sharing a birthday is greater than 50%? How does this compare to your answer to problem 1?

5. Use the method above to determine how many people would need to be in a room so that the probability of two of them having birthdays on the same day of the month is greater than 50%.

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12-5 Enrichment

Binomial Distribution

A binomial distribution is a particular type of probability distribution. To determine a probability a binomial distribution uses the binomial coefficients and the formula ${}_nC_k \cdot\ p^k(1 - p)^{n-k}$ where n is the number of selections, k is the desired outcome, and p is the probability of the desired outcome.

So, if we have a bag that contains 4 red balls and 8 black balls, we can use this formula to determine the probability of drawing exactly 2 red balls out of 5 draws if we return the ball after each draw.

Use the formula to determine the probability: ${}_5C_2\left(\frac{4}{12}\right)^2\left(1 - \frac{4}{12}\right)^3 = \frac{80}{243}$

So, the probability of choosing exactly 2 red balls out of 5 draws is $\frac{80}{243}$.

Exercises

1. Determine the probability of drawing exactly 3 black balls.

2. Compare the probability found in problem 1 to the probability found in the example. Explain why they compare the way they do.

3. Determine the probability of drawing exactly 3 red balls.

4. Determine the probability of drawing exactly 4 red balls.

5. Now assume that there are 3 red balls and 2 black balls in the bag. Complete the probability distribution table using the binomial distribution.

Event	Probability
P(0 Red)	$\frac{8}{125}$
P(1 Red)	
P(2 Red)	
P(3 Red)	

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12-6 Enrichment

Game Shows

On a popular game show of the 70s contestants were presented with three doors. Behind one of the doors was a prize. Behind the other two doors were gag prizes. A contestant was asked to select a door. Instead of showing the contestant what was behind the selected door; the host would reveal a gag prize behind one of the doors that was not chosen. After the gag prize was revealed, the contestant was given the choice to switch doors or stay with the door originally chosen. In this simulation, you will determine whether it is better to stay, switch, or if it does not matter.

1. Make a conjecture as to whether it is better to stay, switch, or does not matter once a gag prize is shown.

2. Find another person to do this simulation with. Use three colored cards to represent the three doors. Two of the cards should be the same color and the third should be a different color. One person should act as the "host" and the other as the "contestant." The "contestant" wins if the uniquely colored card is chosen. The host shuffles the cards and places them color side down in front of the contestant after looking to see which color is where. The contestant then chooses a card and the host reveals one of the gag cards (not chosen by the contestant). The contestant should then decide whether to stay or switch. Record with a tally mark whether the contestant switched or stayed and whether it was a win or loss. Do this 15 times, and then reverse roles.

	Won	Lost
Stay		
Switch		

3. Look at your data and reevaluate. Is it better to stay, switch, or does it matter?

4. What is the probability of being correct the first time? If the contestant did not choose the correct card the first time, the probability of choosing the winning card by switching doors is $\frac{2}{3}$. So, mathemat`ically, is it better to switch or stay? How does this compare to your results?