NAME ____ DATE______________ PERIOD _____

1-1 Enrichment

Traveling on a Budget

Lesson 1-1

You are traveling to your aunt's house 200 miles away for a surprise birthday party. The party starts at 3 P.M. but you cannot leave from your house before 11 A.M. You must fill your gas tank before the trip. Gasoline is \$3.50 per gallon and you have \$30. Will you make it to the party and make it back home?

First determine at which speed you must travel to arrive by 3:00.

1. A simple formula relates the travel time, depending on your average speed in miles per hour (mph), $T = \frac{D}{S}$, where T is time in hours, D is the distance (200 miles), and S is the speed. Determine travel time to your aunt's house at various speeds.

Speed (mph)	Time (hours)
35	
40	
45	
50	
55	
60	
65	
70	
75	

2. For which speed(s), will you miss the surprise birthday party?

Now determine if you can afford enough gasoline to make the trip and return.

3. How many gallons of gas can you buy?

4. Cost depends on the cost of gasoline, the number of total miles of the trip, and your car's fuel efficiency (mi/gal). The miles per gallon can be found using the formula $M = -\frac{1}{30}S^2 + \frac{5}{2}S$, where S is your speed. Determine your fuel rate for the speeds needed to get to your aunt's. Will you make it?

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ____ DATE______________ PERIOD _____

1-2 Enrichment

Properties of a Group

A set of numbers forms a group with respect to an operation if for that operation the set has (1) the Closure Property, (2) the Associative Property, (3) a member which is an identity, and (4) an inverse for each member of the set.

Example 1 **Does the set {0, 1, 2, 3, ...} form a group with respect to addition?**

Closure Property: For all numbers in the set, is $a + b$ in the set? $0 + 1 = 1$, and 1 is in the set; $0 + 2 = 2$, and 2 is in the set; and so on. The set has closure for addition.

Associative Property: For all numbers in the set, does $a + (b + c) = (a + b) + c$? $0 + (1 + 2) = (0 + 1) + 2$; $1 + (2 + 3) = (1 + 2) + 3$; and so on. The set is associative for addition.

Identity: Is there some number, i, in the set such that $i + a = a = a + i$ for all a? $0 + 1 = 1 = 1 + 0$; $0 + 2 = 2 = 2 + 0$; and so on. The identity for addition is 0.

Inverse: Does each number, a, have an inverse, a', such that $a' + a = a + a' = i$? The integer inverse of 3 is -3 since $-3 + 3 = 0$, and 0 is the identity for addition. But the set does not contain -3. Therefore, there is no inverse for 3.

The set is not a group with respect to addition because only three of the four properties hold.

Example 2 **Is the set {−1, 1} a group with respect to multiplication?**

Closure Property: $(-1)(-1) = 1$; $(-1)(1) = -1$; $(1)(-1) = -1$; $(1)(1) = 1$
The set has closure for multiplication.

Associative Property: $(-1)[(-1)(-1)] = (-1)(1) = -1$; and so on
The set is associative for multiplication.

Identity: $1(-1) = -1$; $1(1) = 1$
The identity for multiplication is 1.

Inverse: -1 is the inverse of -1 since $(-1)(-1) = 1$, and 1 is the identity.
1 is the inverse of 1 since $(1)(1) = 1$, and 1 is the identity.
Each member has an inverse.

The set $\{-1, 1\}$ is a group with respect to multiplication because all four properties hold.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

Exercises

Tell whether the set forms a group with respect to the given operation.

1. {integers}, addition

2. {integers}, multiplication

3. $\left\{\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \cdots\right\}$, addition

4. {multiples of 5}, multiplication

5. $\{x, x^2, x^3, x^4, \cdots\}$ addition

6. $\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \cdots\}$, multiplication

7. {irrational numbers}, addition

8. {rational numbers}, addition

1-3 Enrichment

United States' Gross National Product

The Gross National Product, GNP, is an important indicator of U.S. economy. The GNP contains information about the inflation rate, the Bond market, and the Stock market. It is composed of consumer goods, investments, government expenditures, exports, and imports.

Calculated from $GNP = C + I + G + X - M$, where

C is consumer goods (e.g. TV's, Cars, Food, Furniture, Clothes, Doctors' fees, and Dining)

I is investments (e.g. Factories, Computers, Airlines, and Housing)

G is government spending and investments (e.g. Ships, Roads, Schools, NASA, and Bombs)

X is exports (e.g. Corn, Wheat, Cars, and Computers)

M is for imports, (e.g. Cars, Computer chips, Clothes, and Oil)

1. The most important sector of the U.S. economy is consumption. It makes up about 60% of the entire GNP. In 2000, the U.S.'s GNP was 10.5 trillion dollars. In the same year, there were 1 trillion dollars in investments, but a 1 trillion dollar trade deficit. Assuming that consumption made up 60% of the GNP, how much did the government budget for spending?

2. In 2001, the U.S. trade deficit remain at 1 trillion dollars, investments also remain steady at 1 trillion dollars. However consumption dipped to only 50% of the GNP, which increased to 12 trillion dollars. What was the effect on government spending? What might have caused the change?

3. If the GNP remains steady, and so do investments and government spending, but the trade deficit increases (to say 2 or 3 trillion dollars), what does this say about the consumption level?

4. Determine if there is a trade surplus or deficit when there is 12 trillion dollar GNP, 2 trillion in investments, 3 trillion in government investments, and 5 trillion in consumption. Explain why this situation may be favorable.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

1-4 Enrichment

Considering All Cases in Absolute Value Equations

You have learned that absolute value equations with one set of absolute value symbols have two cases that must be considered. For example, $|x + 3| = 5$ must be broken into $x + 3 = 5$ or $-(x + 3) = 5$. For an equation with two sets of absolute value symbols, four cases must be considered.

Consider the problem $|x + 2| + 3 = |x + 6|$. First we must write the equations for the case where $x + 6 \geq 0$ and where $x + 6 < 0$. Here are the equations for these two cases:

$|x + 2| + 3 = x + 6$

$|x + 2| + 3 = -(x + 6)$

Each of these equations also has two cases. By writing the equations for both cases of each equation above, you end up with the following four equations:

$x + 2 + 3 = x + 6$ $\qquad$ $x + 2 + 3 = -(x + 6)$

$-(x + 2) + 3 = x + 6$ $\qquad$ $-x - 2 + 3 = -(x + 6)$

Solve each of these equations and check your solutions in the original equation, $|x + 2| + 3 = |x + 6|$. The only solution to this equation is $-\frac{5}{2}$.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

Exercises

Solve each absolute value equation. Check your solution.

1. $|x - 4| = |x + 7|$

2. $|2x + 9| = |x - 3|$

3. $|-3x - 6| = |5x + 10|$

4. $|x + 4| - 6 = |x - 3|$

5. How many cases would there be for an absolute value equation containing three sets of absolute value symbols?

6. List each case and solve $|x + 2| + |2x - 4| = |x - 3|$. Check your solution.

1-5 Enrichment

Equivalence Relations

A relation R on a set A is an *equivalence relation* if it has the following properties.

Reflexive Property	For any element a of set A, a R a.
Symmetric Property	For all elements a and b of set A, if a R b, then b R a.
Transitive Property	For all elements a, b, and c of set A, if a R b and b R c, then a R c.

Equality on the set of all real numbers is reflexive, symmetric, and transitive. Therefore, it is an equivalence relation.

In each of the following, a relation and a set are given. Write *yes* if the relation is an equivalence relation on the given set. If it is not, tell which of the properties it fails to exhibit.

1. $<$, {all numbers}

2. $\cong$, {all triangles in a plane}

3. is the sister of, {all women in Tennessee}

4. $\geq$, {all numbers}

5. is a factor of, {all nonzero integers}

6. $\sim$, {all polygons in a plane}

7. is the spouse of, {all people in Roanoke, Virginia}

8. $\perp$, {all lines in a plane}

9. is a multiple of, {all integers}

10. is the square of, {all numbers}

11. $\|$, {all lines in a plane}

12. has the same color eyes as, {all members of the Cleveland Symphony Orchestra}

13. is the greatest integer not greater than, {all numbers}

14. is the greatest integer not greater than, {all integers}

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ____ DATE______________ PERIOD _____

1-6 Enrichment

Conjunctions and Disjunctions

An absolute value inequality may be solved as a compound sentence.

Example 1 **Solve $|2x| < 10$.**

$|2x| < 10$ means $2x < 10$ and $2x > -10$.

Solve each inequality. $x < 5$ and $x > -5$.

Every solution for $|2x| < 10$ is a replacement for x that makes both $x < 5$ and $x > -5$ true.

A compound sentence that combines two statements by the word *and* is a *conjunction*.

Example 2 **Solve $|3x - 7| \geq 11$.**

$|3x - 7| \geq 11$ means $3x - 7 \geq 11$ or $3x - 7 \leq -11$.

Solve each inequality. $3x \geq 18$ or $3x \leq -4$

$x \geq 6$ or $x \leq -\frac{4}{3}$

Every solution for the inequality is a replacement for x that makes either $x \geq 6$ or $x \leq -\frac{4}{3}$ true.

A compound sentence that combines two statements by the word *or* is a *disjunction*.

Exercises

Solve each inequality. Then write whether the solution is a conjunction or disjunction.

1. $|4x| > 24$

2. $|x - 7| \leq 8$

3. $|2x + 5| < 1$

4. $|x - 1| \geq 1$

5. $|3x - 1| \leq x$

6. $7 - |2x| > 5$

7. $\left|\frac{x}{2} + 1\right| \geq 7$

8. $\left|\frac{x - 4}{3}\right| < 4$

9. $|8 - x| > 2$

10. $|5 - 2x| \leq 3$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ____ DATE______________ PERIOD _____

2-1 Enrichment

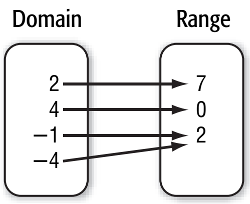
Mappings

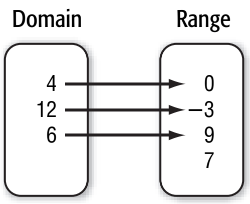
There are three special ways in which one set can be mapped to another.
A mapping can be one-to-one, onto, bijective, or none of these.

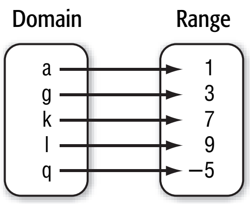
One-to-one mapping	A mapping from set *A* to set *B* where different elements of *A* are never mapped to the same element of *B*.
Onto mapping	A mapping from set *A* to set *B* where each element of set *B* has at least one element of set *A* mapped to it.
Bijective mapping	A mapping from set *A* onto set *B* that is one-to-one and onto.

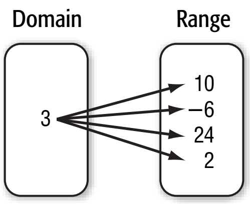
Lesson 2-1

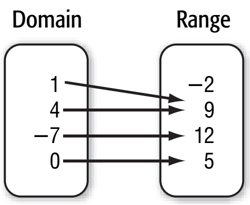
State whether each mapping is one-to-one, onto, bijective, or none of these.

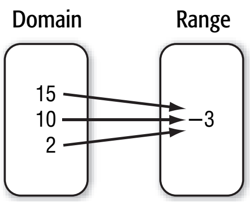
1.


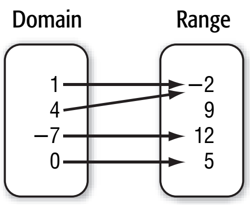
2.


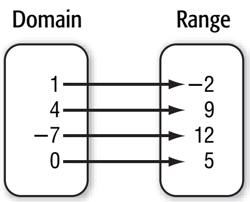
3.


4.


5.


6.


7.


8.


9. Can a set be mapped *onto* a set with fewer elements than it has?

10. Can a set have a one-to-one mapping into a set that has more elements than it has?

11. If a mapping from set *A* into set *B* is bijective, what can you conclude about the number of elements in *A* and *B*?

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

2-2 Enrichment

Diophantine Equations

The first great algebraist, Diophantus of Alexandria (about A.D. 300), devoted much of his work to the solving of indeterminate equations. An indeterminate equation has more than one variable and an unlimited number of solutions. An example is $x + 2y = 4$.

When the coefficients of an indeterminate equation are integers and you are asked to find solutions that must be integers, the equation is called a *diophantine*. Such equations can be quite difficult to solve, often involving trial and error—and some luck!

Solve each diophantine equation by finding at least one pair of positive integers that makes the equation true. Some hints are given to help you.

1. $2x + 5y = 32$

a. First solve the equation for x.

b. Why must y be an even number?

c. Find at least one solution.

2. $5x + 2y = 42$

a. First solve the equation for x.

b. Rewrite your answer in the form $x = 8 +$ some expression.

c. Why must $(2 + 2y)$ be a multiple of 5?

d. Find at least one solution.

3. $2x + 7y = 29$

4. $7x + 5y = 118$

5. $8x - 13y = 100$

6. $3x + 4y = 22$

7. $5x - 14y = 11$

8. $7x + 3y = 40$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

2-3 Enrichment

The Increase in Greenhouse Gases

The atmosphere is composed of about 50% carbon dioxide, CO_2. The levels of carbon dioxide are increasing due to increase fuel consumption and housing and commercial development. The concentration of a compound is measured in parts per million (ppm). For example, if there were 500 CO_2 molecules out of one million air particles, then the CO_2 level would be 500 ppm.

1. In 1965, the concentration of CO_2 was 320 ppm. In 2004, the concentration was 378 ppm. Determine the rate at which CO_2 increased in ppm per year.

2. Carbon dioxide concentration is related to human consumption of fossil fuels and the decrease of trees due to development, therefore an increase in human population will result in an increase in carbon dioxide. In 1980 the U.S. population was 225 million. The 2000 census reported 281 million. At what rate is the population increasing per year? What do you estimate the U.S. population to be today?

3. Use the figures from Exercises 1 and 2 to determine about how much CO_2 is "produced" per million people. Is it possible to reduce the concentration of carbon dioxide in the atmosphere when the human population is increasing? Explain.

4. The greenhouse effect is heat "trapped" by gases such as carbon dioxide, which acts as a "blanket" for the earth. Higher concentration levels of carbon dioxide amplify the greenhouse effect. Thus, global temperature is related to the concentration of CO_2. Records indicate that the increase in global temperature since 1940 is 0.02 degrees Fahrenheit per year. Each degree rise in temperature causes ocean levels to rise one-half a foot. Use your data to determine in what year the ocean level will rise 2 feet. What impact will this have on coastal regions of the United States?

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ______________________________ DATE______________ PERIOD _____

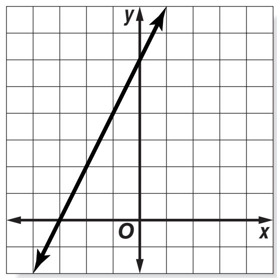
2-4 Enrichment

Two-Intercept Form of a Linear Equation

You are already familiar with the slope-intercept form of a linear equation, $y = mx + b$. Linear equations can also be written in the form $\frac{x}{a} + \frac{y}{b} = 1$ with x-intercept a and y-intercept b. This is called two-intercept form.

Example 1 **Draw the graph of $\frac{x}{-3} + \frac{y}{6} = 1$.**

The graph crosses the x-axis at -3 and the y-axis at 6. Graph $(-3, 0)$ and $(0, 6)$, then draw a straight line through them.



Example 2 **Write $3x + 4y = 12$ in two-intercept form.**

$\frac{3x}{12} + \frac{4y}{12} = \frac{12}{12}$ Divide by 12 to obtain 1 on the right side.

$\frac{x}{4} + \frac{y}{3} = 1$ Simplify.

The x-intercept is 4; the y-intercept is 3.

Exercises

Use the given intercepts *a* and *b*, to write an equation in two-intercept form. Then draw the graph.

1. $a = -2, b = -4$

2. $a = 1, b = 8$

3. $a = 3, b = 5$

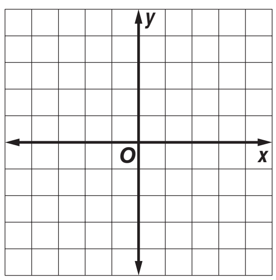
4. $a = 6, b = 9$

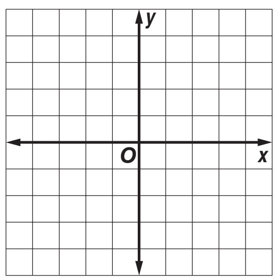
Write each equation in two-intercept form. Then draw the graph.

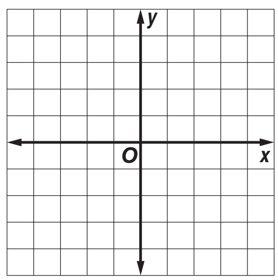
5. $3x - 2y = -6$

6. $\frac{1}{2}x + \frac{1}{4}y = 1$

7. $5x + 2y = -10$







Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

2-5 Enrichment

Median-Fit Lines

A **median-fit line** is a particular type of line of fit. Follow the steps below to find the equation of the median-fit line for the data.

Approximate Percentage of Violent Crimes Committed by Juveniles That Victims Reported to Law Enforcement									
Year	1980	1982	1984	1986	1988	1990	1992	1994	1996
Offenders	36	35	33	32	31	30	29	29	30

Source: U.S. Bureau of Justice Statistics

1. Divide the data into three approximately equal groups. There should always be the same number of points in the first and third groups. In this case, there will be three data points in each group.

2. Find x_1, x_2, and x_3, the medians of the x values in groups 1, 2, and 3, respectively. Find y_1, y_2, and y_3, the medians of the y values in groups 1, 2, and 3, respectively.

3. Find an equation of the line through (x_1, y_1) and (x_3, y_3).

4. Find Y, the y-coordinate of the point on the line in Exercise 2 with an x-coordinate of x_2.

5. The median-fit line is parallel to the line in Exercise 2, but is one-third closer to (x_2, y_2). This means it passes through $\left(x_2, \frac{2}{3}Y + \frac{1}{3}y_2\right)$. Find this ordered pair.

6. Write an equation of the median-fit line.

7. Use the median-fit line to predict the percentage of juvenile violent crime offenders in 2010 and 2020.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ____ DATE______________ PERIOD _____

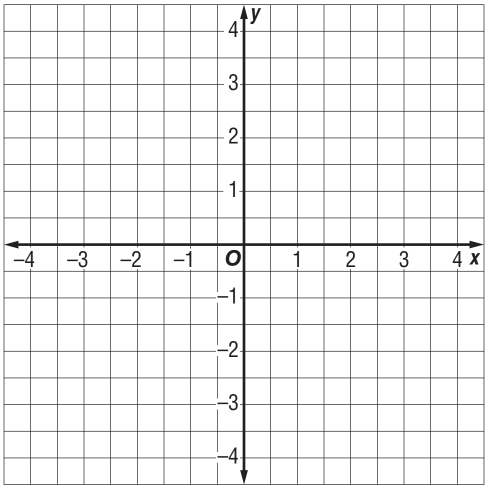
2-6 Enrichment

Graphing Greatest Integer Functions

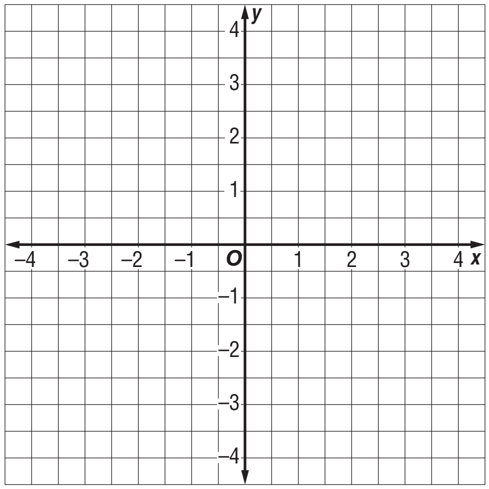
Some equations involving the greatest integer function produce interesting graphs. It will be helpful to make a chart of values for each function and to use a colored pen or pencil.

Graph each function.

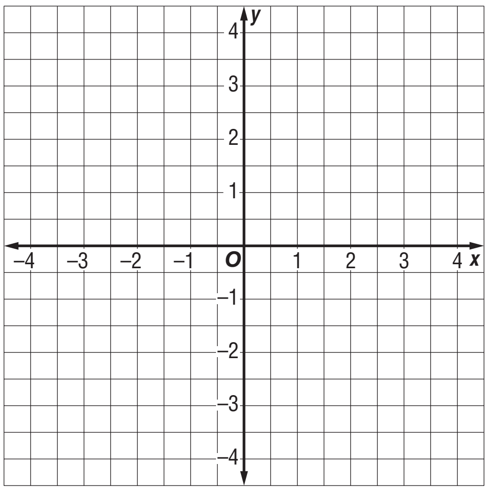
1. $y = 2x - [\![x]\!]$



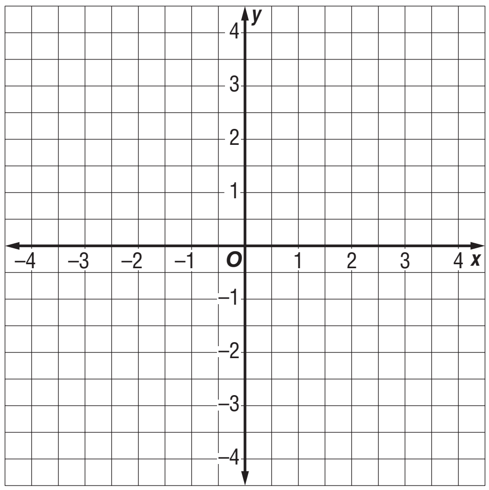
2. $y = \dfrac{[\![x]\!]}{[\![x]\!]}$



3. $y = \dfrac{[\![0.5x + 1]\!]}{[\![0.5x + 1]\!]}$



4. $y = \dfrac{x}{[\![x]\!]}$



Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

2-7 Enrichment

Limits

The concept of the limit is central to many areas of mathematics, especially to calculus. For example, consider the expression, $3x + 2$. As the value of x approaches 1, the value of the expression approaches 5, as shown in the table below. The process of systematically choosing values closer to 1 and producing values closer to 5 demonstrates finding the limit of the expression.

x	$3x + 2$
0.900	4.700
0.950	4.850
0.990	4.970
0.999	4.997
0.9999	4.9997

Find the limits for each expression as *x* approaches the value given.

1. $2x + 2$ as x approaches 5

2. $x - 5$ as x approaches 11

3. $\frac{3x + 5}{x - 6}$ as x approaches 1

4. $\frac{5x - 2}{x + 1}$ as x approaches 1

5. $\frac{3x + 5}{x - 6}$ as x approaches 100

6. $\frac{5x - 2}{x + 1}$ as x approaches 100

7. $\frac{3x + 5}{x - 6}$ as x approaches 1000

8. $\frac{5x - 2}{x + 1}$ as x approaches 1000

9. What do you notice about the limits you found in exercises 3–8?

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

3-1 Enrichment

Solutions to Nonlinear Equations

Real-life situations are often not capable of being represented by a linear equation. Systems of nonlinear equations are often used in the study of population dynamics, modeling carbon monoxide exposure, and determining the height of an object in free fall.

Nonlinear equations have one variable raised to a power other than one or multiplication of two or more variables.

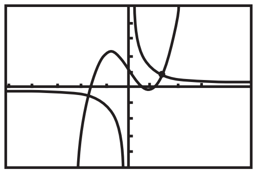
Examples

a. $xy = 1$ or $y = \frac{1}{x}$ — The first equation has a product of two variables. The second is the same equation solved for *y*.

b. $y = x^3 - 2x + 1$ — The variable *x* is raised to the third power.

Systems of nonlinear equations consist of two or more equations, where at least one is nonlinear.

Solutions to these systems are typically difficult to find. One useful method for finding solutions to systems of nonlinear equations is the same as the method for finding solutions to systems of linear equations—use technology to graph the system and find the point(s) of intersection. The graph of the system is shown at the right.



$$\begin{cases} y = \frac{1}{x} \\ y = x^3 - 2x + 1 \end{cases}$$

Using the Intersection function of a graphing calculator you find that one point of intersection is approximately (1.34509, 0.743445).

Exercises

1. Use a graphing calculator to find the other point of intersection.

2. Use the **Zoom** feature on the calculator to zoom in around the point of intersection. What do the two nonlinear equations remind you of at this level of zoom?

3. GOLF The height of a golf ball dropped from the top of a 100-foot tower after t seconds is given by $h = -16t^3 + 100$. Use a graphing calculator to determine when (in seconds) the golf ball is 10 feet from the ground.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

3-2 Enrichment

Using Coordinates

From one observation point, the line of sight to a downed plane is given by $y = x - 1$. This equation describes the distance from the observation point to the plane in a straight line. From another observation point, the line of sight is given by $x + 3y = 21$. What are the coordinates of the point at which the crash occurred?

Solve the system of equations $\begin{cases} y = x - 1 \\ x + 3y = 21 \end{cases}$.

$$\begin{aligned} x + 3y &= 21 \\ x + 3(x - 1) &= 21 \quad \text{Substitute } x - 1 \text{ for } y. \\ x + 3x - 3 &= 21 \\ 4x &= 24 \\ x &= 6 \end{aligned}$$

$$\begin{aligned} x + 3y &= 21 \\ 6 + 3y &= 21 \quad \text{Substitute 6 for } x. \\ 3y &= 15 \\ y &= 5 \end{aligned}$$

The coordinates of the crash are (6, 5).

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

Solve.

1. The lines of sight to a forest fire are as follows.

From Ranger Station A: $3x + y = 9$
From Ranger Station B: $2x + 3y = 13$
Find the coordinates of the fire.

2. An airplane is traveling along the line $x - y = -1$ when it sees another airplane traveling along the line $5x + 3y = 19$. If they continue along the same lines, at what point will their flight paths cross?

3. Two mine shafts are dug along the paths of the following equations.

$x - y = 1400$
$2x + y = 1300$

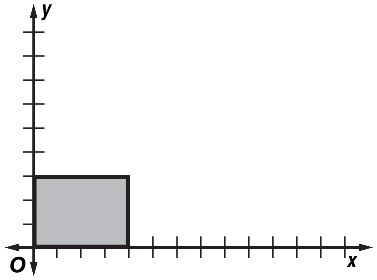
If the shafts meet at a depth of 200 feet, what are the coordinates of the point at which they meet?

3-3 Enrichment

Creative Designs

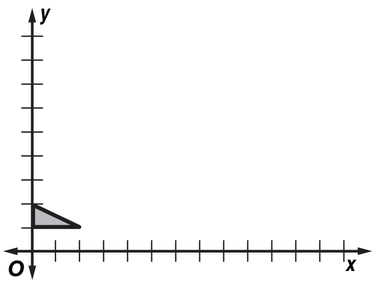
A system of linear inequalities can be used to define the region bounded by a geometric shape graphed on a coordinate plane. For example, the rectangle shown can be defined by the system

$x \leq 4$
$x \geq 0$
$y \leq 3$
$y \geq 0.$

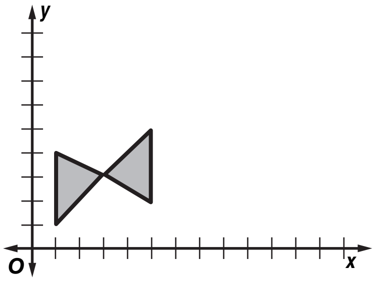


The triangle shown can be described using the inequalities

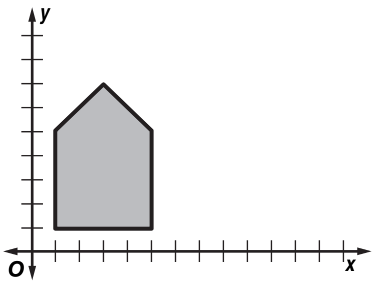
$x + 2y \leq 4$
$x \geq 0$
$y \geq 1.$



1. Find a system of linear inequalities to describe the area bounded by the bow tie shape below. The intersection points are (1, 1), (1, 4), (3, 3), (5, 2), and (5, 5).



2. Find a system of linear inequalities to describe the area bounded by the basic 'house' shape shown below. The intersection points are (1,1), (1,5), (3,7), (5,5), and (5,1).



Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

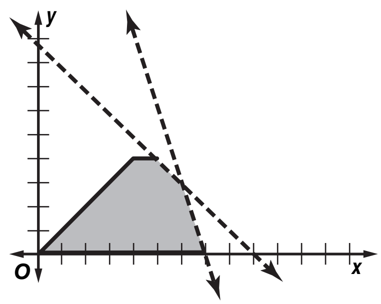
3-4 Enrichment

Sensitivity Analysis

A linear programming model has specific objective coefficients. For example, if the value of a model is found by $2x + 3y = 5$, the objective coefficients are {2, 3}. What if these coefficients were {2.1, 2.9} or {2.5, 3.1}? How would these changes affect the optimal linear program value? This type of investigation is called **sensitivity analysis**.

In general, the objective function in two-variable linear programming problem can be written as: maximize (or minimize) $Ax + By = C$, subject to a set of constraint equations. Changes to the *parameters* A and B could change the slope of the line. This change of slope could lead to a change in the optimum solution to a different corner point (Recall, the optimum solution occurs at a *corner point*).

There is a range in the slope value that will produce this change, thus there is a range of variation for both A and B that will keep the optimal solution the same (see graph).



1. Find the slope of $Ax + By = C$ and observe how changes to the parameters A and B can change the slope of the line.

Consider the Linear Programming problem:

Maximize: $C = 2x + 3y$,
Subject to: $3x + y \le 21$
$x + y \le 9$
$y \le x$
$y \le 4$

(x, y)	(0, 0)	(4, 4)	(5, 4)	(6, 3)	(7, 0)
C	0	20	22	21	14

After finding the intersections and evaluating the objective equation, we find the maximal solution is (5, 4). If the objective coefficients are changed from 2 and 3 to A and B, the optimum solution with remain at (5, 4) while the slope remains between the slope of $x + y \le 9$ and the slope of $3x + y \le 21$. If not, then the new optimal solution will be at (4, 4) or (6, 3).

2. Express the relationship, the slope of the objective function is between the slope of the line $x + y = 9$ and the slope of the line $3x + y = 21$, algebraically.

3. Determine the range on A if B remains equal to 3.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

Lesson 3-4

3-5 Enrichment

Homogenous Systems

A system of equations is called homogeneous if it is of the form:

$$gz + hy + kz = 0$$
$$dx + ey + fz = 0$$
$$ax + by + cz = 0$$

Homogeneous systems have some unique characteristics that set them apart from general systems of equations. The following exercises will explore some of these unique characteristics.

1. Evaluate the following statement. Is this statement *always*, *sometimes*, or *never* true? Expalin your reasoning.
Every homogeneous system of equations will have at least one trivial solution: (0, 0, 0).

2. Find a non-trivial solution to the following homogenous system of equations.

$$x + y + 5z = 0$$
$$2x + y + 7z = 0$$
$$x + 2z = 0$$

3. Multiply the solution you found in Exercise 2 by 3. Is the new ordered triple a solution to the system?

4. Multiply the solution you found in Exercise 2 by -6. Is the new ordered triple a solution to the system?

5. Make a conjecture about any multiple of a given solution to a homogeneous system of equations.

6. Make a conjecture about the number of solutions that a homogeneous system of equations will have if it has at least one non-trivial solution.

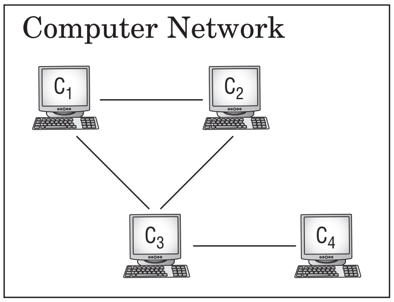
Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

4-1 Enrichment

Matrices and Networks

Graph theory is a branch of mathematics that explores situations represented by points called vertices and the segments that may connect them, called edges. For example, graphs can be used to represent computer networks or airline routes between major cities.

An incidence matrix is a matrix used to represent the vertices, edges, and relationships among the vertices of a graph. The vertices name each row and column. For example, the incidence matrix for the computer network shown in the figure is shown below. The numbers represent how many edges connect the vertices.



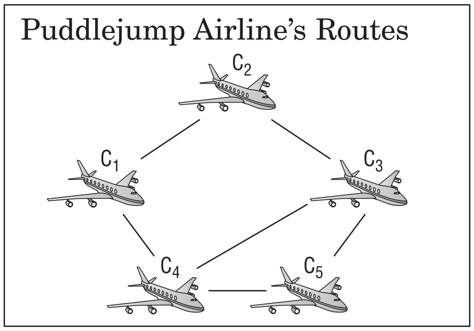
Indicates one edge from C_1 to C_2.

	C_1	C_2	C_3	C_4
C_1	0	1	1	0
C_2	1	0	1	0
C_3	1	1	0	1
C_4	0	0	1	0

Indicates no edge from C_4 to C_1.

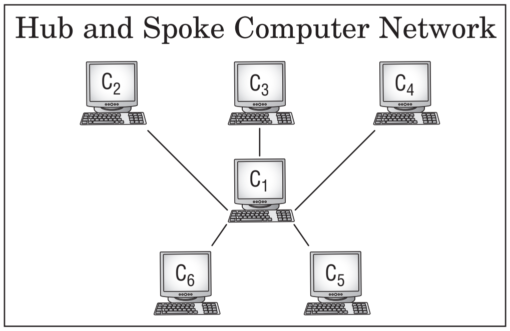
Complete the incidence matrix for each pictured network.

1. Puddlejump Airlines daily flights between cities 1–5.



	c_1	c_2	c_3	c_4	c_5
c_1	___	___	___	___	___
c_2	___	___	___	___	___
c_3	___	___	___	___	___
c_4	___	___	___	___	___
c_5	___	___	___	___	___

2. Computers in a Hub and Spoke network.



	c_1	c_2	c_3	c_4	c_5	c_6
c_1	___	___	___	___	___	___
c_2	___	___	___	___	___	___
c_3	___	___	___	___	___	___
c_4	___	___	___	___	___	___
c_5	___	___	___	___	___	___
c_6	___	___	___	___	___	___

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

4-2 Enrichment

Sundaram's Sieve

The properties and patterns of prime numbers have fascinated many mathematicians. In 1934, a young East Indian student named Sundaram constructed the following matrix.

4	7	10	13	16	19	22	25	. . .
7	12	17	22	27	32	37	42	. . .
10	17	24	31	38	45	52	59	. . .
13	22	31	40	49	58	67	76	. . .
16	27	38	49	60	71	82	93	. . .
.	.	.	.	.	.	.	.	. . .

A surprising property of this matrix is that it can be used to determine whether or not some numbers are prime.

Complete these problems to discover this property.

1. The first row and the first column are created by using an arithmetic pattern. What is the common difference used in the pattern?

2. Find the next four numbers in the first row.

3. What are the common differences used to create the patterns in rows 2, 3, 4, and 5?

4. Write the next two rows of the matrix. Include eight numbers in each row.

5. Choose any five numbers from the matrix. For each number n, that you chose from the matrix, find $2n + 1$.

6. Write the factorization of each value of $2n + 1$ that you found in problem 5.

7. Use your results from problems 5 and 6 to complete this statement: If n occurs in the matrix, then $2n + 1$ ____________ (is/is not) a prime number.

8. Choose any five numbers that are not in the matrix. Find $2n + 1$ for each of these numbers. Show that each result is a prime number.

9. Complete this statement: If n does not occur in the matrix, then $2n + 1$ is ____________________.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

4-3 Enrichment

Properties of Matrices

Computing with matrices is different from computing with real numbers. Stated below are some properties of the real number system. Are these also true for matrices? In the problems on this page, you will investigate this question.

For all real numbers a and b, $ab = 0$ if and only if $a = 0$ or $b = 0$.

Multiplication is commutative. For all real numbers a and b, $ab = ba$.

Multiplication is associative. For all real numbers a, b, and c, $a(bc) = (ab)c$.

Use the matrices *A*, *B*, and *C* for the problems. Write whether each statement is true. Assume that a 2-by-2 matrix is the 0 matrix if and only if all of its elements are zero.

$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix}$ $C = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$

1. $AB = 0$

2. $AC = 0$

3. $BC = 0$

4. $AB = BA$

5. $AC = CA$

6. $BC = CB$

7. $A(BC) = (AB)C$

8. $B(CA) = (BC)A$

9. $B(AC) = (BA)C$

10. Write a statement summarizing your findings about the properties of matrix multiplication.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

Lesson 4-3

NAME ____ DATE______________ PERIOD _____

4-4 Enrichment

Computer Graphics Using Matrix Transformations

Computer animation creates a sensation of movement by slowly changing the position of pixels of an image on a two-dimensional video screen. By selecting the center of the screen as the origin, each pixel can be located using ordered pairs. A series of delayed matrix transformations (transformation, pause, transformation, pause) applied to the coordinates of the point (pixel) produces the desired animation effect.

Two computer programmers, Nate and Daniel, are writing a computer animation program using only the two matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Nate argues that it does not make a difference which matrix is applied first. Daniel disagrees.

1. What transformation does each matrix represent?

2. Find AB and BA. Determine whether Nate or Daniel is correct.

3. A third programmer joins Nate and Daniel. She convinces them to expand their transformation capabilities by including a reflection about the y-axis using $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

a. Determine a transformation using only B and C that is equivalent to the transformation A.

b. Write an expression for a 180° counterclockwise rotation in terms of the reflections A and C.

c. Is it possible to express a 270° counterclockwise rotation in terms of the reflections A and C? Explain why or why not.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ____ DATE______________ PERIOD _____

4-5 Enrichment

Matrix Transpose and Determinants

In Lesson 4-1, you learned how to represent information in matrices. A matrix contains elements of the form a_{ij} where i is the row number of the element and j is the columns number of the element.

Consider the following matrix.

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$$

In this matrix, $a_{11} = 2$, $a_{12} = -1$, $a_{21} = 3$, and $a_{22} = 4$.

The matrix transpose can be found by switching the elements around. Element a_{ij} becomes element a_{ji}. So, the matrix transpose of A, denoted by A^T is:

$$A^T = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$$

Calculate the determinant of A and A^T.

$\det(A) = 2(4) - 3(-1) = 11$

$\det(A^T) = 2(4) - (-1)(3) = 11$

1. Find each matrix transpose.

a. $B = \begin{bmatrix} -1 & 5 \\ 2 & 6 \end{bmatrix}$ **b.** $C = \begin{bmatrix} 1 & 0 \\ -3 & 4 \end{bmatrix}$ **c.** $D = \begin{bmatrix} 2 & 3 & -1 \\ -2 & -1 & 5 \\ 1 & 3 & -2 \end{bmatrix}$

2. Find the determinants of the original matrices and the transposes from Exercise 1.

3. What do you notice about the determinants? Make a conjecture about the determinant of a matrix and the determinant of its transpose.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

4-6 Enrichment

Fourth-Order Determinants

To find the value of a 4 × 4 determinant, use a method called **expansion by minors**.

First write the expansion. Use the first row of the determinant.
Remember that the signs of the terms alternate.

$$\begin{vmatrix} 6 & -3 & 2 & 7 \\ 0 & 4 & 3 & 5 \\ 0 & 2 & 1 & -4 \\ 6 & 0 & -2 & 0 \end{vmatrix} = 6\begin{vmatrix} 4 & 3 & 5 \\ 2 & 1 & -4 \\ 0 & -2 & 0 \end{vmatrix} - (-3)\begin{vmatrix} 0 & 3 & 5 \\ 0 & 1 & -4 \\ 6 & -2 & 0 \end{vmatrix} + 2\begin{vmatrix} 0 & 4 & 5 \\ 0 & 2 & -4 \\ 6 & 0 & 0 \end{vmatrix} - 7\begin{vmatrix} 0 & 4 & 3 \\ 0 & 2 & 1 \\ 6 & 0 & -2 \end{vmatrix}$$

Then evaluate each 3 × 3 determinant. Use any row.

$$\begin{vmatrix} 4 & 3 & 5 \\ 2 & 1 & -4 \\ 0 & -2 & 0 \end{vmatrix} = -(-2)\begin{vmatrix} 4 & 5 \\ 2 & -4 \end{vmatrix}$$
$$= 2(-16 - 10)$$
$$= -52$$

$$\begin{vmatrix} 0 & 3 & 5 \\ 0 & 1 & -4 \\ 6 & -2 & 0 \end{vmatrix} = -3\begin{vmatrix} 0 & -4 \\ 6 & 0 \end{vmatrix} + 5\begin{vmatrix} 0 & 1 \\ 6 & -2 \end{vmatrix}$$
$$= -3(24) + 5(-6)$$
$$= -102$$

$$\begin{vmatrix} 0 & 4 & 5 \\ 0 & 2 & -4 \\ 6 & 0 & 0 \end{vmatrix} = 6\begin{vmatrix} 4 & 5 \\ 2 & -4 \end{vmatrix}$$
$$= 6(-16 - 10)$$
$$= -156$$

$$\begin{vmatrix} 0 & 4 & 3 \\ 0 & 2 & 1 \\ 6 & 0 & -2 \end{vmatrix} = -4\begin{vmatrix} 0 & 1 \\ 6 & -2 \end{vmatrix} + 3\begin{vmatrix} 0 & 2 \\ 6 & 0 \end{vmatrix}$$
$$= -4(-6) + 3(-12)$$
$$= -12$$

Finally, evaluate the original 4 × 4 determinant.

$$\begin{vmatrix} 6 & -3 & 2 & 7 \\ 0 & 4 & 3 & 5 \\ 0 & 2 & 1 & -4 \\ 6 & 0 & -2 & 0 \end{vmatrix} = 6(-52) + 3(-102) + 2(-156) - 7(-12) = -846$$

Evaluate each determinant.

1. $\begin{vmatrix} 1 & 2 & 3 & 1 \\ 4 & 3 & -1 & 0 \\ 2 & -5 & 4 & 4 \\ 1 & -2 & 0 & 2 \end{vmatrix}$

2. $\begin{vmatrix} 3 & 3 & 3 & 3 \\ 2 & 1 & 2 & 1 \\ 4 & 3 & -1 & 5 \\ 2 & 5 & 0 & 1 \end{vmatrix}$

3. $\begin{vmatrix} 1 & 4 & 3 & 0 \\ -2 & -3 & 6 & 4 \\ 5 & 1 & 1 & 2 \\ 4 & 2 & 5 & -1 \end{vmatrix}$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

4-7 Enrichment

Permutation Matrices

A permutation matrix is a square matrix in which each row and each column has one entry that is 1. All the other entries are 0. Find the inverse of a permutation matrix interchanging the rows and columns. For example, row 1 is interchanged with column 1, row 2 is interchanged with column 2.

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

P is a 4×4 permutation matrix. P^{-1} is the inverse of P.

Solve each problem.

1. There is just one 2×2 permutation matrix that is not also an identity matrix. Write this matrix.

2. Find the inverse of the matrix you wrote in Exercise 1. What do you notice?

3. Show that the two matrices in Exercises 1 and 2 are inverses.

4. Write the inverse of this matrix.

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

5. Use B^{-1} from problem 4. Verify that B and B^{-1} are inverses.

6. Permutation matrices can be used to write and decipher codes. To see how this is done, use the message matrix M and matrix B from problem 4. Find matrix C so that C equals the product MB. Use the rules below.

0 times a letter = 0

1 times a letter = the same letter

0 plus a letter = the same letter

$$M = \begin{bmatrix} \text{S} & \text{H} & \text{E} \\ \text{S} & \text{A} & \text{W} \\ \text{H} & \text{I} & \text{M} \end{bmatrix}$$

7. Now find the product CB^{-1}. What do you notice?

Lesson 4-7

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

4-8 Enrichment

Determining Political Popularity

Systems of equations have applications in branches of science including chemistry, ecology, and physics. They can also be used to describe situations involving social studies and politics.

Consider the two candidates for City Council, Jefferson Dailey and Robert Jackson. Support for a candidate is measured by a positive number less than 1 and opposition of a candidate by a negative number greater than −1. For example, 0.75 indicates fairly high support, while −0.75 means fairly high opposition. Simultaneous support for, or opposition to, both candidates is possible. Generally, however, if one candidate is popular and is supported while the other candidate is opposed, support of the popular candidate tends to decrease as support for the "underdog" rises. Let the change in support for Jefferson Dailey be denoted by ΔJ(delta J) and the change in support for Robert Jackson is denoted by ΔR (delta R).

This situation is described by the system of equations:

$$\begin{cases} \Delta J = -0.5J + 0.25R \\ \Delta R = -0.25J - 0.5R \end{cases}$$

For example, if $\Delta J = -0.2$ and $\Delta R = 0.2$, then current support for Jefferson Daily is decreasing at a rate of 20% while Robert Jackson's support is increasing at 20%. Substituting the given values for ΔJ and ΔR and solving the first equation for R yields:

$$R = \frac{0.5J - 0.2}{0.25}$$

Substituting this expression for R in the second equation and solving for J yields:

$$0.2 = 0.25J - 0.5\left(\frac{0.5J - 0.2}{0.25}\right) \Rightarrow J = 0.267$$

Therefore, $R = -0.267$.

If the election were held today, Jefferson Daily would win.

Solve the systems of equations for the following values of ΔJ and ΔR to determine potential winners and losers.

1. $\Delta J = -0.1$, $\Delta R = 0.2$

2. $\Delta J = 0.5$, $\Delta R = -0.1$

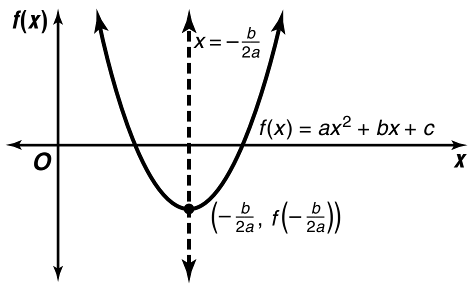
Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

5-1 Enrichment

Finding the Axis of Symmetry of a Parabola

As you know, if $f(x) = ax^2 + bx + c$ is a quadratic function, the values of x that make $f(x)$ equal to zero are $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

The average of these two number values is $-\frac{b}{2a}$. The function $f(x)$ has its maximum or minimum value when $x = -\frac{b}{2a}$. Since the axis of symmetry of the graph of $f(x)$ passes through the point where the maximum or minimum occurs, the axis of symmetry has the equation $x = -\frac{b}{2a}$.



Example **Find the vertex and axis of symmetry for $f(x) = 5x^2 + 10x - 7$.**

Use $x = -\frac{b}{2a}$.

$x = -\frac{10}{2(5)} = -1$ The x-coordinate of the vertex is -1.

Substitute $x = -1$ in $f(x) = 5x^2 + 10x - 7$.

$f(-1) = 5(-1)^2 + 10(-1) - 7 = -12$

The vertex is $(-1, -12)$.

The axis of symmetry is $x = -\frac{b}{2a}$, or $x = -1$.

Exercises

Find the vertex and axis of symmetry for the graph of each function using $x = -\frac{b}{2a}$.

1. $f(x) = x^2 - 4x - 8$

2. $g(x) = -4x^2 - 8x + 3$

3. $y = -x^2 + 8x + 3$

4. $f(x) = 2x^2 + 6x + 5$

5. $A(x) = x^2 + 12x + 36$

6. $k(x) = -2x^2 + 2x - 6$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ____ DATE______________ PERIOD _____

5-2 Enrichment

Graphing Absolute Value Equations

You can solve absolute value equations in much the same way you solved quadratic equations. Graph the related absolute value function for each equation using a graphing calculator. Then use the ZERO feature in the CALC menu to find its real solutions, if any. Recall that solutions are points where the graph intersects the x-axis.

For each equation, make a sketch of the related graph and find the solutions rounded to the nearest hundredth.

1. $|x + 5| = 0$

2. $|4x - 3| + 5 = 0$

3. $|x - 7| = 0$

4. $|x + 3| - 8 = 0$

5. $-|x + 3| + 6 = 0$

6. $|x - 2| - 3 = 0$

7. $|3x + 4| = 2$

8. $|x + 12| = 10$

9. $|x| - 3 = 0$

10. Explain how solving absolute value equations algebraically and finding zeros of absolute value functions graphically are related.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

5-3 Enrichment

Using Patterns to Factor

Study the patterns below for factoring the sum and the difference of cubes.

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

This pattern can be extended to other odd powers. Study these examples.

Example 1 **Factor $a^5 + b^5$.**

Extend the first pattern to obtain $a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$.

Check: $(a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4) = a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4$
$\qquad + a^4b - a^3b^2 + a^2b^3 - ab^4 + b^5$
$\qquad = a^5 + b^5$

Example 2 **Factor $a^5 - b^5$.**

Extend the second pattern to obtain $a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$.

Check: $(a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4) = a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4$
$\qquad - a^4b - a^3b^2 - a^2b^3 - ab^4 - b^5$
$\qquad = a^5 - b^5$

In general, if n is an odd integer, when you factor $a^n + b^n$ or $a^n - b^n$, one factor will be either $(a + b)$ or $(a - b)$, depending on the sign of the original expression. The other factor will have the following properties:

- The first term will be a^{n-1} and the last term will be b^{n-1}.
- The exponents of a will decrease by 1 as you go from left to right.
- The exponents of b will increase by 1 as you go from left to right.
- The degree of each term will be $n - 1$.
- If the original expression was $a^n + b^n$, the terms will alternately have + and − signs.
- If the original expression was $a^n - b^n$, the terms will all have + signs.

Use the patterns above to factor each expression.

1. $a^7 + b^7$
2. $c^9 - d^9$
3. $e^{11} + f^{11}$

To factor $x^{10} - y^{10}$, change it to $(x^5 + y^5)(x^5 - y^5)$ and factor each binomial. Use this approach to factor each expression.

4. $x^{10} - y^{10}$

5. $a^{14} - b^{14}$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ____ DATE______________ PERIOD _____

5-4 Enrichment

Conjugates and Absolute Value

When studying complex numbers, it is often convenient to represent a complex number by a single variable. For example, we might let $z = x + y\boldsymbol{i}$. We denote the conjugate of z by $\bar{z}$. Thus, $\bar{z} = x - y\boldsymbol{i}$.

We can define the absolute value of a complex number as follows.

$|z| = |x + y\boldsymbol{i}| = \sqrt{x^2 + y^2}$

There are many important relationships involving conjugates and absolute values of complex numbers.

Example 1 **Show $|z|^2 = z\bar{z}$ for any complex number z.**

Let $z = x + y\boldsymbol{i}$. Then,

$$\begin{aligned} z\bar{z} &= (x + y\boldsymbol{i})(x - y\boldsymbol{i}) \\ &= x^2 + y^2 \\ &= \sqrt{(x^2 + y^2)^2} \\ &= |z|^2 \end{aligned}$$

Example 2 **Show $\frac{\bar{z}}{|z|^2}$ is the multiplicative inverse for any nonzero complex number z.**

We know $|z|^2 = z\bar{z}$. If $z \neq 0$, then we have $z\left(\frac{\bar{z}}{|z|^2}\right) = 1$.

Thus, $\frac{\bar{z}}{|z|^2}$ is the multiplicative inverse of z.

Exercises

For each of the following complex numbers, find the absolute value and multiplicative inverse.

1. $2\boldsymbol{i}$

2. $-4 - 3\boldsymbol{i}$

3. $12 - 5\boldsymbol{i}$

4. $5 - 12\boldsymbol{i}$

5. $1 + \boldsymbol{i}$

6. $\sqrt{3} - \boldsymbol{i}$

7. $\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3}\boldsymbol{i}$

8. $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\boldsymbol{i}$

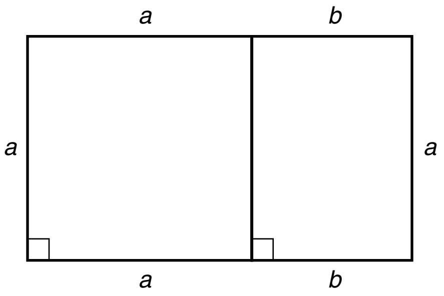
9. $\frac{1}{2} - \frac{\sqrt{3}}{2}\boldsymbol{i}$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

5-5 Enrichment

The Golden Quadratic Equations

A **golden rectangle** has the property that its length can be written as $a + b$, where a is the width of the rectangle and $\frac{a + b}{a} = \frac{a}{b}$. Any golden rectangle can be divided into a square and a smaller golden rectangle, as shown.



The proportion used to define golden rectangles can be used to derive two quadratic equations. These are sometimes called *golden quadratic equations*.

Solve each problem.

1. In the proportion for the golden rectangle, let a equal 1. Write the resulting quadratic equation and solve for b.

2. In the proportion, let b equal 1. Write the resulting quadratic equation and solve for a.

3. Describe the difference between the two golden quadratic equations you found in exercises 1 and 2.

4. Show that the positive solutions of the two equations in exercises 1 and 2 are reciprocals.

5. Use the Pythagorean Theorem to find a radical expression for the diagonal of a golden rectangle when $a = 1$.

6. Find a radical expression for the diagonal of a golden rectangle when $b = 1$.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

5-6 Enrichment

Sum and Product of Roots

Sometimes you may know the roots of a quadratic equation without knowing the equation itself. Using your knowledge of factoring to solve an equation, you can work backward to find the quadratic equation. The rule for finding the sum and product of roots is as follows:

Sum and Product of Roots	If the roots of $ax^2 + bx + c = 0$, with $a \neq 0$, are s_1 and s_2, then $s_1 + s_2 = -\frac{b}{a}$ and $s_1 \cdot s_2 = \frac{c}{a}$.

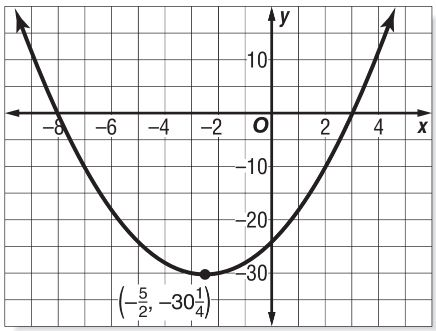
Example **A road with an initial gradient, or slope, of 3% can be represented by the formula $y = ax^2 + 0.03x + c$, where y is the elevation and x is the distance along the curve. Suppose the elevation of the road is 1105 feet at points 200 feet and 1000 feet along the curve. You can find the equation of the transition curve. Equations of transition curves are used by civil engineers to design smooth and safe roads.**

The roots are $x = 3$ and $x = -8$.

$3 + (-8) = -5$ Add the roots.

$3(-8) = -24$ Multiply the roots.

Equation: $x^2 + 5x - 24 = 0$



Exercises

Write a quadratic equation that has the given roots.

1. 6, −9

2. 5, −1

3. 6, 6

4. $4 \pm \sqrt{3}$

6. $-\frac{2}{5}, \frac{2}{7}$

6. $\frac{-2 \pm 3\sqrt{5}}{7}$

Find k such that the number given is a root of the equation.

7. $7;\ 2x^2 + kx - 21 = 0$

8. $-2;\ x^2 - 13x + k = 0$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

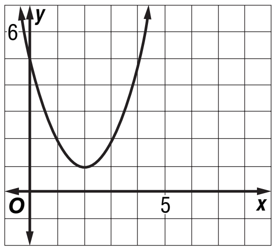
5-7 Enrichment

A Shortcut to Complex Roots

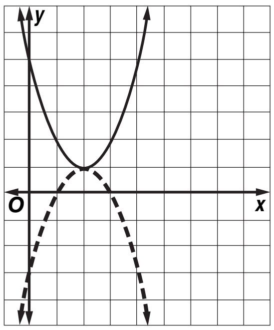
When graphing a quadratic function, the real roots are shown in the graph. You have learned that quadratic functions can also have imaginary roots that cannot be seen on the graph of the function. However, there is a way to graphically represent the complex roots of a quadratic function.

Example **Find the complex roots of the quadratic function $y = x^2 - 4x + 5$.**

Step 1: Graph the function.



Step 2: Reflect the graph over the horizontal line containing the vertex. In this example, the vertex is (2, 1).



Step 3: The real part of the complex root is the point halfway between the x-intercepts of the reflected graph and the imaginary part of the complex roots are + and − half the distance between the x-intercepts of the reflected graph. So, in this example, the complex roots are $2 + 1i$ and $2 - 1i$.

Exercises

Using this method, find the complex roots of the following quadratic functions.

1. $y = x^2 + 2x + 5$

2. $y = x^2 + 4x + 8$

3. $y = x^2 + 6x + 13$

4. $y = x^2 + 2x + 17$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

5-8 Enrichment

Graphing Absolute Value Inequalities

You can solve absolute value inequalities by graphing in much the same manner you graphed quadratic inequalities. Graph the related absolute function for each inequality by using a graphing calculator. For $>$ and $\geq$, identify the x-values, if any, for which the graph lies *below* the x-axis. For $<$ and $\leq$, identify the x values, if any, for which the graph lies *above* the x-axis.

For each inequality, make a sketch of the related graph and find the solutions rounded to the nearest hundredth.

1. $|x - 3| > 0$

2. $|x| - 6 < 0$

3. $-|x + 4| + 8 < 0$

4. $2|x + 6| - 2 \geq 0$

5. $|3x - 3| \geq 0$

6. $|x - 7| < 5$

7. $|7x - 1| > 13$

8. $|x - 3.6| \leq 4.2$

9. $|2x + 5| \leq 7$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

6-1 Enrichment

Properties of Exponents

The rules about powers and exponents are usually given with letters such as m, n, and k to represent exponents. For example, one rule states that $a^m \cdot a^n = a^{m+n}$.

In practice, such exponents are handled as algebraic expressions and the rules of algebra apply.

Example 1 **Simplify $2a^2(a^{n+1} + a^{4n})$.**

$2a^2(a^{n+1} + a^{4n}) = 2a^2 \cdot a^{n+1} + 2a^2 \cdot a^{4n}$ Use the Distributive Law.

$= 2a^{2+n+1} + 2a^{2+4n}$ Recall $a^m \cdot a^n = a^{m+n}$.

$= 2a^{n+3} + 2a^{2+4n}$ Simplify the exponent $2 + n + 1$ as $n + 3$.

It is important always to collect *like* terms only.

Example 2 **Simplify $(a^n + b^m)^2$.**

$(a^n + b^m)^2 = (a^n + b^m)(a^n + b^m)$

F O I L

$= a^n \cdot a^n + a^n \cdot b^m + a^n \cdot b^m + b^m \cdot b^m$ The second and third terms are like terms.

$= a^{2n} + 2a^nb^m + b^{2m}$

Exercises

Simplify each expression by performing the indicated operations.

1. $2^3 2^m$

2. $(a^3)^n$

3. $(4^nb^2)^k$

4. $(x^3a^j)^m$

5. $(-ay^n)^3$

6. $(-b^kx)^2$

7. $(c^2)^{hk}$

8. $(-2d^n)^5$

9. $(a^2b)(a^nb^2)$

10. $(x^ny^m)(x^my^n)$

11. $\frac{a^n}{a^2}$

12. $\frac{12x^3}{4x^n}$

13. $(ab^2 - a^2b)(3a^n + 4b^n)$

14. $ab^2(2a^2b^{n-1} + 4ab^n + 6b^{n+1})$

Lesson 6-1

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ____ DATE______________ PERIOD _____

6-2 Enrichment

Polynomials with Fractional Coefficients

Polynomials may have fractional coefficients as long as there are no variables in the denominators. Computing with fractional coefficients is performed in the same way as computing with whole-number coefficients.

Simpliply. Write all coefficients as fractions.

1. $\left(\frac{3}{5}m - \frac{2}{7}p - \frac{1}{3}n\right) - \left(\frac{7}{3}p - \frac{5}{2}m - \frac{3}{4}n\right)$

2. $\left(\frac{3}{2}x - \frac{4}{3}y - \frac{5}{4}z\right) + \left(-\frac{1}{4}x + y + \frac{2}{5}z\right) + \left(-\frac{7}{8}x - \frac{6}{7}y + \frac{1}{2}z\right)$

3. $\left(\frac{1}{2}a^2 - \frac{1}{3}ab + \frac{1}{4}b^2\right) + \left(\frac{5}{6}a^2 + \frac{2}{3}ab - \frac{3}{4}b^2\right)$

4. $\left(\frac{1}{2}a^2 - \frac{1}{3}ab + \frac{1}{4}b^2\right) - \left(\frac{1}{3}a^2 - \frac{1}{2}ab + \frac{5}{6}b^2\right)$

5. $\left(\frac{1}{2}a^2 - \frac{1}{3}ab + \frac{1}{4}b^2\right) \cdot \left(\frac{1}{2}a - \frac{2}{3}b\right)$

6. $\left(\frac{2}{3}a^2 - \frac{1}{5}a + \frac{2}{7}\right) \cdot \left(\frac{2}{3}a^3 + \frac{1}{5}a^2 - \frac{2}{7}a\right)$

7. $\left(\frac{2}{3}x^2 - \frac{3}{4}x - 2\right) \cdot \left(\frac{4}{5}x - \frac{1}{6}x^2 - \frac{1}{2}\right)$

8. $\left(\frac{1}{6} + \frac{1}{3}x + \frac{1}{6}x^4 - \frac{1}{2}x^2\right) \cdot \left(\frac{1}{6}x^3 - \frac{1}{3} - \frac{1}{3}x\right)$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

6-3 Enrichment

Oblique Asymptotes

The graph of $y = ax + b$, where $a \neq 0$, is called an oblique asymptote of $y = f(x)$ if the graph of f comes closer and closer to the line as $x \to \infty$ or $x \to -\infty$. ∞ is the mathematical symbol for **infinity**, which means *endless*.

For $f(x) = 3x + 4 + \frac{2}{x}$, $y = 3x + 4$ is an oblique asymptote because $f(x) - 3x - 4 = \frac{2}{x}$, and $\frac{2}{x} \to 0$ as $x \to \infty$ or $-\infty$. In other words, as $|x|$ increases, the value of $\frac{2}{x}$ gets smaller and smaller approaching 0.

Example **Find the oblique asymptote for $f(x) = \frac{x^2 + 8x + 15}{x + 2}$.**

$$\begin{array}{r|rrr} -2 & 1 & 8 & 15 \\ & & -2 & -12 \\ \hline & 1 & 6 & 3 \end{array}$$

Use synthetic division.

$$y = \frac{x^2 + 8x + 15}{x + 2} = x + 6 + \frac{3}{x + 2}$$

As $|x|$ increases, the value of $\frac{3}{x + 2}$ gets smaller. In other words, since $\frac{3}{x + 2} \to 0$ as $x \to \infty$ or $x \to -\infty$, $y = x + 6$ is an oblique asymptote.

Exercises

Use synthetic division to find the oblique asymptote for each function.

1. $y = \frac{8x^2 - 4x + 11}{x + 5}$

2. $y = \frac{x^2 + 3x - 15}{x - 2}$

3. $y = \frac{x^2 - 2x - 18}{x - 3}$

4. $y = \frac{ax^2 + bx + c}{x - d}$

5. $y = \frac{ax^2 + bx + c}{x + d}$

Lesson 6-3

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

6-4 Enrichment

Approximation by Means of Polynomials

Many scientific experiments produce pairs of numbers $[x, f(x)]$ that can be related by a formula. If the pairs form a function, you can fit a polynomial to the pairs in exactly one way. Consider the pairs given by the following table.

x	1	2	4	7
$f(x)$	6	11	39	−54

We will assume the polynomial is of degree three. Substitute the given values into this expression.

$$f(x) = A + B(x - x_0) + C(x - x_0)(x - x_1) + D(x - x_0)(x - x_1)(x - x_2)$$

You will get the system of equations shown below. You can solve this system and use the values for $A, B, C,$ and D to find the desired polynomial.

$$6 = A$$
$$11 = A + B(2 - 1) = A + B$$
$$39 = A + B(4 - 1) + C(4 - 1)(4 - 2) = A + 3B + 6C$$
$$-54 = A + B(7 - 1) + C(7 - 1)(7 - 2) + D(7 - 1)(7 - 2)(7 - 4) = A + 6B + 30C + 90D$$

Solve.

1. Solve the system of equations for the values $A, B, C,$ and D.

2. Find the polynomial that represents the four ordered pairs. Write your answer in the form $y = a + bx + cx^2 + dx^3$.

3. Find the polynomial that gives the following values.

x	8	12	15	20
$f(x)$	−207	169	976	3801

4. A scientist measured the volume $f(x)$ of carbon dioxide gas that can be absorbed by one cubic centimeter of charcoal at pressure x. Find the values for $A, B, C,$ and D.

x	120	340	534	698
$f(x)$	3.1	5.5	7.1	8.3

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

6-5 Enrichment

Golden Rectangles

Use a straightedge, a compass, and the instructions below to construct a golden rectangle.

1. Construct square *ABCD* with sides of 2 centimeters.

2. Construct the midpoint of $\overline{AB}$. Call the midpoint *M*.

3. Using *M* as the center, set your compass opening at *MC*. Construct an arc with center *M* that intersects $\overline{AB}$. Call the point of intersection *P*.

4. Construct a line through *P* that is perpendicular to $\overline{AB}$.

5. Extend $\overline{DC}$ so that it intersects the perpendicular. Call the intersection point *Q*. *APQD* is a golden rectangle. Check this conclusion by finding the value of $\frac{QP}{AP}$.

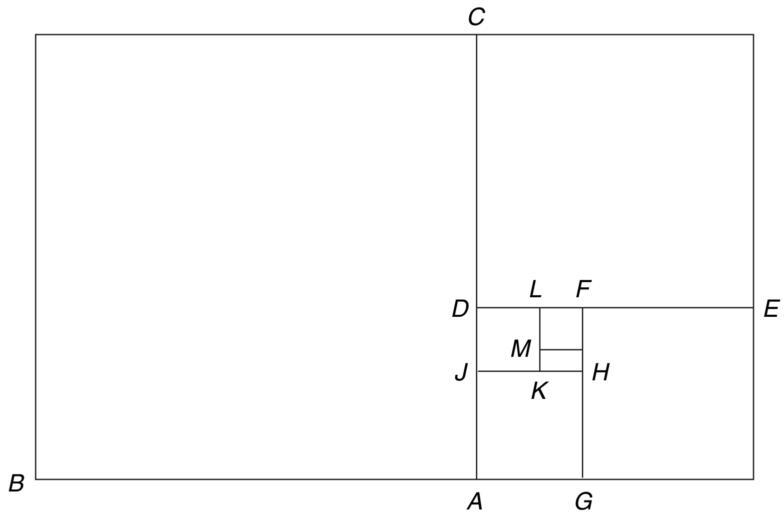
A figure consisting of similar golden rectangles is shown below. Use a compass and the instructions below to draw quarter-circle arcs that form a spiral like that found in the shell of a chambered nautilus.

6. Using *A* as a center, draw an arc that passes through *B* and *C*.

7. Using *D* as a center, draw an arc that passes through *C* and *E*.

8. Using *F* as a center, draw an arc that passes through *E* and *G*.

9. Continue drawing arcs, using *H*, *K*, and *M* as the centers.



Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

6-6 Enrichment

History of Quadratic Equations

The ancient Babylonians are believed to be the first to solve quadratic equations, around 400 B.C. Euclid, who devised a geometrical approach in 300 B.C., followed them. Around 598–665 A.D., a Hindu mathematician named Brahmagupta created an almost modern method for solving equations. Finally, around 800 A.D., an Arab mathematician named al-Khwarizmi created a classification of quadratic equations. He classified them into six different categories and devoted a chapter to each type. His equations are made up of three different types of expressions: roots (x), squares of roots (x^2) and numbers.

For example, his first classification was squares equal to roots. A sample of this type of equations is: $x^2 = 2x$.

Now solve this quadratic equation.

$x^2 = 2x$	
$x^2 - 2x = 0$	Subtract $2x$ from each side.
$x(x - 2) = 0$	Factor.
$x = 0$ or $x - 2 = 0$	Set both factors equal to 0.
So, $x = 0$ or 2	Solve.

Write and solve a sample problem for the remaining 5 classifications of quadratic equations, according to al-Khwarizmi.

1. Squares equal to numbers.

2. Roots equal to numbers.

3. Squares and roots equal to numbers.

4. Squares and numbers equal to roots.

5. Roots and numbers equal to squares.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

6-7 Enrichment

Radical Notation

In 1494, the first Edition of *Summa de arithmetica geometrica proprtioni et proportionalita*, now known as the *Suma*, was printed in Italy. The author, Luca Pacioli, wrote the book as a summary of the mathematical knowledge at the time. However, the notation used in the book is quite similar to the notation used today. For example, to represent radicals, the following was used:

$6 \,.\, p \,.\, R \,.\, 10$

In our notation, the p represents "plus" and the R represents "radical." So, $6 \,.\, p \,.\, R \,.\, 10$ means $6 + \sqrt{10}$.

1. What letter would you expect to represent subtraction?

2. Translate the following notations into modern notation.

a. $18 \,.\, m \,.\, R \,.\, 90$

b. $108 \,.\, m \,.\, R \,.\, 3240 \,.\, p \,.\, R \,.\, 3240 \,.\, m \,.\, R \,.\, 900$

c. $10 \,.\, R \,.\, 5 \,.\, p \,.\, 2 \,.\, R \,.\, 3$

3. Translate the following into notations from 1494.

a. $32\sqrt{10}$

b. $21\sqrt{6} + 3\sqrt{3}$

c. $5\sqrt{2} - 2 + 7\sqrt{11}$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

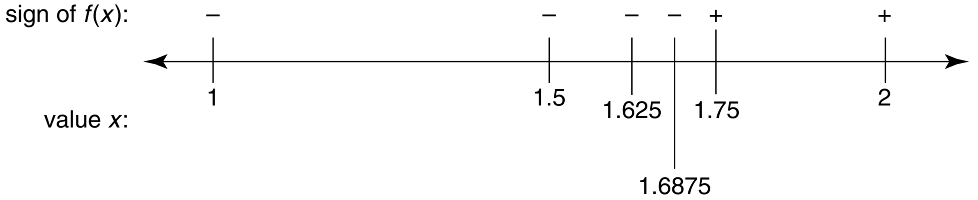
6-8 Enrichment

The Bisection Method for Approximating Real Zeros

The **bisection method** can be used to approximate zeros of polynomial functions like $f(x) = x^3 + x^2 - 3x - 3$.

Since $f(1) = -4$ and $f(2) = 3$, there is at least one real zero between 1 and 2. The midpoint of this interval is $\frac{1 + 2}{2} = 1.5$. Since $f(1.5) = -1.875$, the zero is between 1.5 and 2. The midpoint of this interval is $\frac{1.5 + 2}{2} = 1.75$. Since $f(1.75)$ is about 0.172, the zero is between 1.5 and 1.75. The midpoint of this interval is $\frac{1.5 + 1.75}{2} = 1.625$ and $f(1.625)$ is about -0.94. The zero is between 1.625 and 1.75. The midpoint of this interval is $\frac{1.625 + 1.75}{2} = 1.6875$. Since $f(1.6875)$ is about -0.41, the zero is between 1.6875 and 1.75. Therefore, the zero is 1.7 to the nearest tenth.

The diagram below summarizes the results obtained by the bisection method.



Using the bisection method, approximate to the nearest tenth the zero between the two integral values of *x* for each function.

1. $f(x) = x^3 - 4x^2 - 11x + 2, f(0) = 2, f(1) = -12$

2. $f(x) = 2x^4 + x^2 - 15, f(1) = -12, f(2) = 21$

3. $f(x) = x^5 - 2x^3 - 12, f(1) = -13, f(2) = 4$

4. $f(x) = 4x^3 - 2x + 7, f(-2) = -21, f(-1) = 5$

5. $f(x) = 3x^3 - 14x^2 - 27x + 126, f(4) = -14, f(5) = 16$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

6-9 Enrichment

Irrational Numbers

Philosopher Hippasus of Metapontum was believed to have discovered that $\sqrt{2}$ was irrational. Mathematicians of the time denied the existence of irrational numbers and killed Hippasus, not wishing to believe this fundamental number could fail to be a ratio of integers.

The typical way to prove that $\sqrt{2}$ is irrational is by contradiction and relies on a few other common facts that are easily proven. That is, the proof assumes that it is rational and deduces a contradiction.

Theorem: $\sqrt{2}$ is irrational

Proof: Suppose $\sqrt{2}$ is a rational number. Then $\sqrt{2} = \frac{a}{b}$, where a and b are relatively prime integers. Relatively prime integers are integers that have no common factor other than one, therefore $\frac{a}{b}$ is a fraction written in lowest terms. It is also this condition that provides the contradiction. If we square both sides of the equation, $\sqrt{2} = \frac{a}{b}$, we have $2 = \frac{a^2}{b^2}$. This is equivalent to $a^2 = 2b^2$. However, this says that a^2 is an even number, thus a is an even number. If a is even, b is also even. Thus a and b have a factor in common other than one, namely two, and are not relatively prime. Hence $\sqrt{2}$ is irrational.

The Rational Zero Theorem provides a direct proof method.

Exercises

1. Use the rational zero theorem to prove that $\sqrt{2}$ is irrational.

2. Show that the square of an even number is even.

3. Show that any integer zeros of a polynomial function must be factors of the constant term a_0.

Lesson 6-9

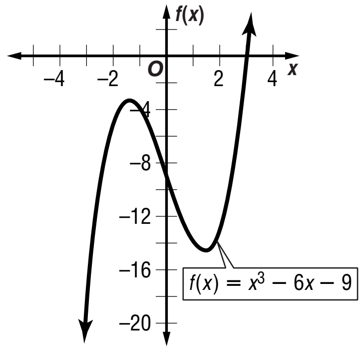
Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

7-1 Enrichment

Relative Maximum Values

The graph of $f(x) = x^3 - 6x - 9$ shows a relative maximum value somewhere between $f(-2)$ and $f(-1)$. You can obtain a closer approximation by comparing values such as those shown in the table.

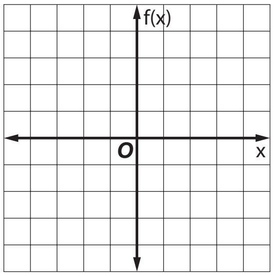
To the nearest tenth a relative maximum value for $f(x)$ is -3.3.



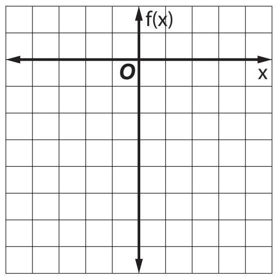
x	$f(x)$
−2	−5
−1.5	−3.375
−1.4	−3.344
−1.3	−3.397
−1	−4

Using a calculator to find points, graph each function. To the nearest tenth, find a relative maximum value of the function.

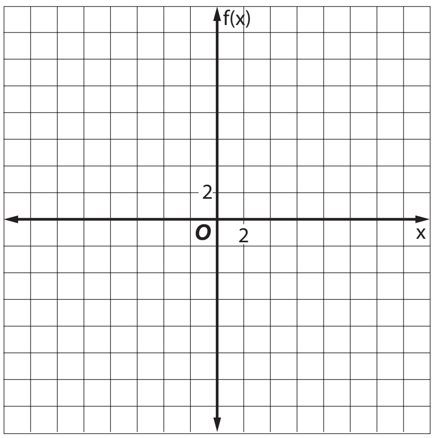
1. $f(x) = x(x^2 - 3)$



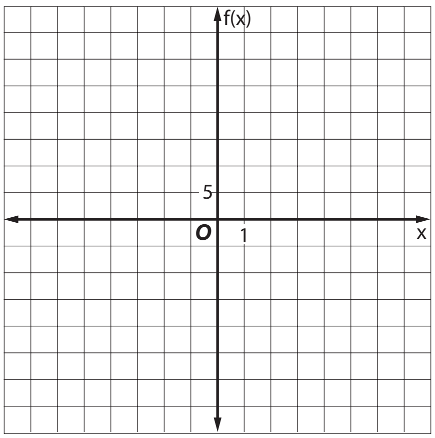
2. $f(x) = x^3 - 3x - 3$



3. $f(x) = x^3 - 9x - 2$



4. $f(x) = x^3 + 2x^2 - 12x - 24$



Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

7-2 Enrichment

Reading Algebra

In mathematics, the term *group* has a special meaning. The following numbered sentences discuss the idea of group and one interesting example of a group.

01 To be a group, a set of elements and a binary operation must satisfy four conditions: the set must be closed under the operation, the operation must be associative, there must be an identity element, and every element must have an inverse.

02 The following six functions form a group under the operation of composition of functions: $f_1(x) = x$, $f_2(x) = \frac{1}{x}$, $f_3(x) = 1 - x$, $f_4(x) = \frac{(x-1)}{x}$, $f_5(x) = \frac{x}{(x-1)}$, and $f_6(x) = \frac{1}{(1-x)}$.

03 This group is an example of a noncommutative group. For example, $f_3 \circ f_2 = f_4$, but $f_2 \circ f_3 = f_6$.

04 Some experimentation with this group will show that the identity element is f_1.

05 Every element is its own inverse except for f_4 and f_6, each of which is the inverse of the other.

Use the paragraph to answer these questions.

1. Explain what it means to say that a set is *closed* under an operation. Is the set of positive integers closed under subtraction?

2. Subtraction is a noncommutative operation for the set of integers. Write an informal definition of noncommutative.

3. For the set of integers, what is the identity element for the operation of multiplication? Justify your answer.

4. Explain how the following statement relates to sentence 05:

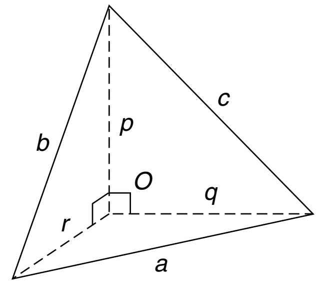
$$(f_6 \cdot f_4)(x) = f_6[f_4(x)] = f_6\left(\frac{1}{(1-x)}\right) = \frac{1}{\frac{1-(x-1)}{x}} = x = f_1(x).$$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

7-3 Enrichment

Reading Algebra

If two mathematical problems have basic structural similarities, they are said to be **analogous.** Using analogies is one way of discovering and proving new theorems.



The following numbered sentences discuss a three-dimensional analogy to the Pythagorean theorem.

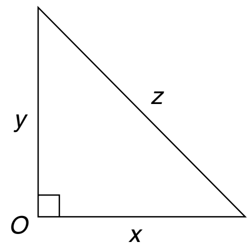
01 Consider a tetrahedron with three perpendicular faces that meet at vertex *O*.

02 Suppose you want to know how the areas *A, B,* and *C* of the three faces that meet at vertex *O* are related to the area *D* of the face opposite vertex *O*.

03 It is natural to expect a formula analogous to the Pythagorean theorem $z^2 = x^2 + y^2$, which is true for a similar situation in two dimensions.

04 To explore the three-dimensional case, you might guess a formula and then try to prove it.

05 Two reasonable guesses are $D^3 = A^3 + B^3 + C^3$ and $D^2 = A^2 + B^2 + C^2$.



Refer to the numbered sentences to answer the questions.

1. Use sentence 01 and the top diagram. The prefix *tetra-* means four. Write an informal definition of tetrahedron.

2. Use sentence 02 and the top diagram. What are the lengths of the sides of each face of the tetrahedron?

3. Rewrite sentence 01 to state a two-dimensional analogue.

4. Refer to the top diagram and write expressions for the areas *A, B,* and *C* mentioned in sentence 02.

5. To explore the three-dimensional case, you might begin by expressing *a, b,* and *c* in terms of *p, q,* and *r.* Use the Pythagorean theorem to do this.

6. Which guess in sentence 05 seems more likely? Justify your answer.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME _______________________________ DATE_____________ PERIOD _____

7-4 Enrichment

Approximating Square Roots

Consider the following expansion.

$$\left(a + \frac{b}{2a}\right)^2 = a^2 + \frac{2ab}{2a} + \frac{b^2}{4a^2}$$

$$= a^2 + b + \frac{b^2}{4a^2}$$

Think what happens if a is very great in comparison to b. The term $\frac{b^2}{4a^2}$ is very small and can be disregarded in an approximation.

$$\left(a + \frac{b}{2a}\right)^2 \approx a^2 + b$$

$$a + \frac{b}{2a} \approx \sqrt{a^2 + b}$$

Suppose a number can be expressed as $a^2 + b$, $a > b$. Then an approximate value of the square root is $a + \frac{b}{2a}$. You should also see that $a - \frac{b}{2a} \approx \sqrt{a^2 - b}$.

Example **Use the formula $\sqrt{a^2 \pm b} \approx a \pm \frac{b}{2a}$ to approximate $\sqrt{101}$ and $\sqrt{622}$.**

a. $\sqrt{101} = \sqrt{100 + 1} = \sqrt{10^2 + 1}$

Let $a = 10$ and $b = 1$.

$\sqrt{101} \approx 10 + \frac{1}{2(10)}$

≈ 10.05

b. $\sqrt{622} = \sqrt{625 - 3} = \sqrt{25^2 - 3}$

Let $a = 25$ and $b = 3$.

$\sqrt{622} \approx 25 - \frac{3}{2(25)}$

≈ 24.94

Exercises

Use the formula to find an approximation for each square root to the nearest hundredth. Check your work with a calculator.

1. $\sqrt{626}$

2. $\sqrt{99}$

3. $\sqrt{402}$

4. $\sqrt{1604}$

5. $\sqrt{223}$

6. $\sqrt{80}$

7. $\sqrt{4890}$

8. $\sqrt{2505}$

9. $\sqrt{3575}$

10. $\sqrt{1{,}441{,}100}$

11. $\sqrt{290}$

12. $\sqrt{260}$

13. Show that $a - \frac{b}{2a} \approx \sqrt{a^2 - b}$ for $a > b$.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

7-5 Enrichment

Special Products with Radicals

Notice that $(\sqrt{3})(\sqrt{3}) = 3$, or $(\sqrt{3})^2 = 3$. In general, $(\sqrt{x})^2 = x$ when $x \geq 0$.

Also, notice that $(\sqrt{9})(\sqrt{4}) = \sqrt{36}$. In general, $(\sqrt{x})(\sqrt{y}) = \sqrt{xy}$ when x and y are not negative.

You can use these ideas to find the special products below.

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b$$
$$(\sqrt{a} + \sqrt{b})^2 = (\sqrt{a})^2 + 2\sqrt{ab} + (\sqrt{b})^2 = a + 2\sqrt{ab} + b$$
$$(\sqrt{a} - \sqrt{b})^2 = (\sqrt{a})^2 - 2\sqrt{ab} + (\sqrt{b})^2 = a - 2\sqrt{ab} + b$$

Example 1 **Find the product: $(\sqrt{2} + \sqrt{5})(\sqrt{2} - \sqrt{5})$.**

$(\sqrt{2} + \sqrt{5})(\sqrt{2} - \sqrt{5}) = (\sqrt{2})^2 - (\sqrt{5})^2 = 2 - 5 = -3$

Example 2 **Evaluate $(\sqrt{2} + \sqrt{8})^2$.**

$(\sqrt{2} + \sqrt{8})^2 = (\sqrt{2})^2 + 2\sqrt{2}\sqrt{8} + (\sqrt{8})^2$

$= 2 + 2\sqrt{16} + 8 = 2 + 2(4) + 8 = 2 + 8 + 8 = 18$

Exercises

Multiply.

1. $(\sqrt{3} - \sqrt{7})(\sqrt{3} + \sqrt{7})$

2. $(\sqrt{10} + \sqrt{2})(\sqrt{10} - \sqrt{2})$

3. $(\sqrt{2x} - \sqrt{6})(\sqrt{2x} + \sqrt{6})$

4. $(\sqrt{3} - (-7))^2$

5. $(\sqrt{1000} + \sqrt{10})^2$

6. $(\sqrt{y} + \sqrt{5})(\sqrt{y} - \sqrt{5})$

7. $(\sqrt{50} - \sqrt{x})^2$

8. $(\sqrt{x} + 20)^2$

You can extend these ideas to patterns for sums and differences of cubes. Study the pattern below. Then complete Exercises 9–12.

$(\sqrt[3]{8} - \sqrt[3]{x})(\sqrt[3]{8^2} + \sqrt[3]{8x} + \sqrt[3]{x^2}) = \sqrt[3]{8^3} - \sqrt[3]{x^3} = 8 - x$

9. $(\sqrt[3]{2} - \sqrt[3]{5})(\sqrt[3]{2^2} + \sqrt[3]{10} + \sqrt[3]{5^2})$

10. $(\sqrt[3]{y} + \sqrt[3]{w})(\sqrt[3]{y^2} - \sqrt[3]{yw} + \sqrt[3]{w^2})$

11. $(\sqrt[3]{7} + \sqrt[3]{20})(\sqrt[3]{7^2} - \sqrt[3]{140} + \sqrt[3]{20^2})$

12. $(\sqrt[3]{11} - \sqrt[3]{8})(\sqrt[3]{11^2} + \sqrt[3]{88} + \sqrt[3]{8^2})$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

7-6 Enrichment

Lesser-Known Geometric Formulas

Many geometric formulas involve radical expressions.

Make a drawing to illustrate each of the formulas given on this page. Then evaluate the formula for the given value of the variable. Round answers to the nearest hundredth.

1. The area of an isosceles triangle. Two sides have length a; the other side has length c. Find A when $a = 6$ and $c = 7$.

$A = \frac{c}{4}\sqrt{4a^2 - c^2}$

2. The area of an equilateral triangle with a side of length a. Find A when $a = 8$.

$A = \frac{a^2}{4}\sqrt{3}$

3. The area of a regular pentagon with a side of length a. Find A when $a = 4$.

$A = \frac{a^2}{4}\sqrt{25 + 10\sqrt{5}}$

4. The area of a regular hexagon with a side of length a. Find A when $a = 9$.

$A = \frac{3a^2}{2}\sqrt{3}$

5. The volume of a regular tetrahedron with an edge of length a. Find V when $a = 2$.

$V = \frac{a^3}{12}\sqrt{2}$

6. The area of the curved surface of a right cone with an altitude of h and radius of base r. Find S when $r = 3$ and $h = 6$.

$S = \pi r\sqrt{r^2 + h^2}$

7. Heron's Formula for the area of a triangle uses the semi-perimeter s, where $s = \frac{a + b + c}{2}$. The sides of the triangle have lengths a, b, and c. Find A when $a = 3$, $b = 4$, and $c = 5$.

$A = \sqrt{s(s - a)(s - b)(s - c)}$

8. The radius of a circle inscribed in a given triangle also uses the semi-perimeter. Find r when $a = 6$, $b = 7$, and $c = 9$.

$r = \frac{\sqrt{s(s - a)(s - b)(s - c)}}{s}$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ______________________________ DATE______________ PERIOD _____

7-7 Enrichment

Lesson 7-7

Truth Tables

In mathematics, the basic operations are addition, subtraction, multiplication, division, finding a root, and raising to a power. In logic, the basic operations are the following: *not* ($\sim$), *and* ($\wedge$), *or* ($\vee$), and *implies* ($\rightarrow$).

If p and q are statements, then $\sim p$ means not p; $\sim q$ means not q; $p \wedge q$ means p and q; $p \vee q$ means p or q; and $p \rightarrow q$ means p implies q. The operations are defined by truth tables. On the left below is the truth table for the statement $\sim p$. Notice that there are two possible conditions for p, true (T) or false (F). If p is true, $\sim p$ is false; if p is false, $\sim p$ is true. Also shown are the truth tables for $p \wedge q$, $p \vee q$, and $p \rightarrow q$.

p	$\sim p$
T	F
F	T

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

You can use this information to find out under what conditions a complex statement is true.

Example **Under what conditions is $\sim p \vee q$ true?**

Create the truth table for the statement. Use the information from the truth table above for $p \vee q$ to complete the last column.

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

The truth table indicates that $\sim p \vee q$ is true in all cases except where p is true and q is false.

Use truth tables to determine the conditions under which each statement is true.

1. $\sim p \vee \sim q$

2. $\sim p \rightarrow (p \rightarrow q)$

3. $(p \vee q) \vee (\sim p \wedge \sim q)$

4. $(p \rightarrow q) \vee (q \rightarrow p)$

5. $(p \rightarrow q) \wedge (q \rightarrow p)$

6. $(\sim p \wedge \sim q) \rightarrow \sim(p \vee q)$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

8-1 Enrichment

Dimensional Analysis

Scientists always express the units of measurement in their solution. It is insufficient and ambiguous to state a solution regarding distance as 17; Seventeen what, feet, miles, meters? Often it is helpful to analyze the units of the quantities in a formula to determine the desired units of an output. For example, it is known that torque is the product of force and distance, but what are the units of force?

The units also depend on the measuring system. The two most commonly used systems are the British system and the international system of units (SI). Some common units of the British system are inches, feet, miles, and pounds. Common SI units include meters, kilometers, Newtons, and grams. Frequently conversion from one system to another is necessary and accomplished by multiplication by conversion factors.

Consider changing units from miles per hour to kilometers per hour. What is 60 miles per hour in kilometers per hour? Use the conversion 1 ft = 30.5 cm.

$$60\,\frac{\text{mi}}{\text{h}} = 60\,\frac{\text{mi}}{\text{h}} \times \frac{5280\text{ ft}}{1\text{ mi}} \times \frac{30.5\text{ cm}}{1\text{ ft}} \times \frac{1\text{ m}}{100\text{ cm}} \times \frac{1\text{ km}}{1000\text{ m}} = 96.62\,\frac{\text{km}}{\text{h}}$$

1. The SI unit for force is a Newton (N) and the SI unit for distance is meters or centimeters. The British unit for force is pounds and the British unit for distance is feet or inches. Using the formula for torque (Torque = Force times Distance), determine the SI unit and the British unit for torque.

2. The density of a fluid is given by the formula $density = \frac{mass}{volume}$. Suppose that a volume of a fluid in a cylindrical can is $\pi r^2 h$, where r and h are measured in meters. Find an expression for the mass, given in kilograms (kg), of gasoline, which has a known density of $680\,\frac{kg}{m^3}$.

3. Convert the following measurements.

a. 72 miles/hour to feet/second

b. 32 pounds/square inch to pounds per square foot

c. 100 kilometers/hour to miles per hour

Lesson 8-1

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ____ DATE______________ PERIOD _____

8-2 Enrichment

Zeno's Paradox

The Greek philosopher Zeno of Elea (born sometime between 495 and 480 B.C.) proposed four paradoxes to challenge the notions of space and time. Zeno's first paradox works like this:

Suppose you are on your way to school. Assume you are able to cover half of the remaining distance each minute that you walk. You leave your house at 7:45 A.M. After the first minute, you are half of the way to school. In the next minute you cover half of the remaining distance to school, and at 7:47 A.M. you are three-quarters of the way to school. This pattern continues each minute. At what time will you arrive at school? Before 8:00 A.M.? Before lunch?

Since space is infinitely divisible, we can repeat this pattern forever. Thus, on the way to school you must reach an infinite number of 'midpoints' in a finite time. This is impossible, so you can never reach your goal. In general, according to Zeno anyone who wants to move from one point to another must meet these requirements, and motion is impossible. Therefore, what we perceive as motion is merely an illusion.

Addition of fractions can be defined by $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$, similarly for subtraction.

Assume your house is one mile from school. At 7:46 A.M., you have walked half of a mile, so you have left $1 - \frac{1}{2}$, or $\frac{1}{2}$ a mile. At 7:47 A.M. you only have $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ of a mile to go.

To determine how far you have walked and how far away from the school you are at 7:48 A.M., add the distances walked each minute, $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$ of a mile so far and you still have $1 - \frac{7}{8} = \frac{1}{8}$ of a mile to go.

1. Determine how far you have walked and how far away from the school you are at 7:50 A.M.

2. Suppose instead of covering one-half the distance to school each minute, you cover three-quarters of the distance remaining to school each minute, now will you be able to make it to school on time? Determine how far you still have left to go at 7:47 A.M.

3. Suppose that instead of covering one-half or three-quarters of the distance to school each minute, you cover $\frac{1}{x + 1}$ of the distance remaining, where x is a whole number greater than 2. What is your distance from school at 7:46 A.M.?

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME _______________________________________ DATE_____________ PERIOD _____

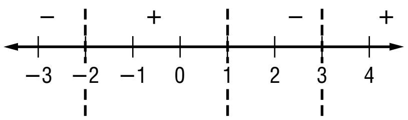
8-3 Enrichment

Characteristics of Rational Function Graphs

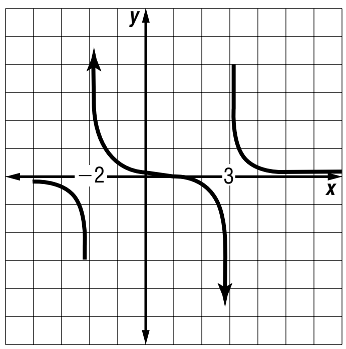
Use the information in the table to graph rational functions

CHARACTERISTIC	MEANING	HOW TO FIND IT
Vertical asymptotes	A vertical line at an *x* value where the rational function is undefined	Set the denominator equal to zero and solve for *x*.
Horizontal asymptotes	A horizontal line that the rational function	Study the end-behaviors.
Right end-behavior	How the graph behaves at large positive values of *x*	Evaluate the rational expression at increasing positive values of *x*.
Left end-behavior	How the graph behaves at large negative values of *x*	Evaluate the rational expression at increasing negative values of *x*.
Roots, zeros, or *x*-intercepts	Point(s) where the graph crosses the *x*-axis	Set the numerator equal to zero and solve for *x*.
***y*-intercepts**	Point where the graph crosses the *y*-axis	Set $x = 0$ to determine the *y*-intercept.

Example A **sign chart** uses an *x* value from the left and right of each critical value to determine if the graph is positive or negative on that interval. A sign chart for $y = \frac{x-1}{x^2-x-6}$ is shown below.

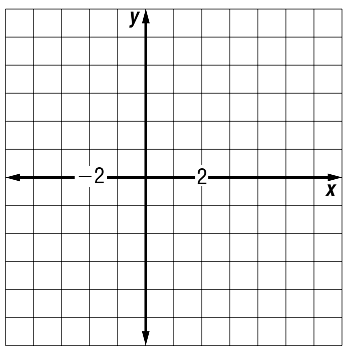


The graph of $\frac{x-1}{x^2-x-6}$ is shown to the right.



Exercise

Create a sign chart for $y = \frac{x-1}{x^2-4}$. Use an *x*-value from the left and right of each critical value to determine if the graph is positive or negative on that interval. Then graph the function.



Lesson 8-3

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

8-4 Enrichment

Geosynchronous Satellites

Satellites circling the Earth are almost as common as the cell phones that depend on them. A geosynchronous satellite is one that maintains the same position above the Earth at all times. Geosynchronous satellites are used in cell phone communications, transmitting signals from towers on Earth and to each other.

The speed at which they travel is very important. If the speed is too low, the satellite will be forced back down to Earth due to the Earth's gravity. However, if it is too fast, it will overcome gravity's force and escape into space, never to return. Newton's second law of motion says that force on an object is equal to mass times acceleration or $F = ma$. It is also well known that the net gravitational force between two objects is inversely proportional to the square of the distance between them. Therefore, there are two variables on which the force depends: speed and height above the Earth.

In particular, Newton's second law, $F = ma$, shows that force varies directly with acceleration, where m is the constant taking the place of "k."

Exercises

1. Show that the net gravitational force providing a satellite with acceleration is inversely proportional to the square of the distance between them by expressing this variation as an equation.

2. Use your equation from Number 1 and equate it with Newton's formula above to determine how the satellite's acceleration varies with its height above the Earth.

3. Determine how the speed of a geosynchronous satellite varies with its height above the Earth by using the fact that speed is equal to distance divided by time and the path of the satellite is circular.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

8-5 Enrichment

Physical Properties of Functions

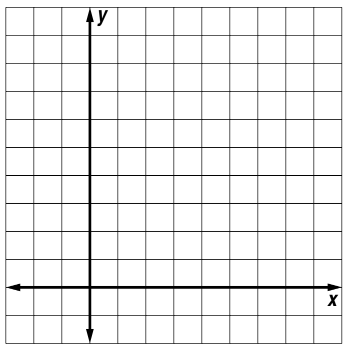
Mathematical functions are classified based on properties similar to how biologists classify animal species. Functions can be classified as continuous or non-continuous, increasing or decreasing, polynomial or non-polynomial for example. The class of polynomials functions can be further classified as linear, quadratic, cubic, etc., based on its *degree*.

Characteristics of functions include:

- A function is **bounded below** if there exists a number that is less than any function value.
- A function is **bounded above** if a number exists that is greater than any function value.
- A function is **symmetric** (about a vertical axis) if it is a mirror image about that vertical axis.
- A function is **continuous** if it can be drawn without lifting your pencil.
- A function is **increasing** if $f(x) > f(y)$ when $x > y$. Continual growth from left to right.
- A function is **decreasing** if $f(x) < f(y)$ when $x < y$. Continual decay from left to right.

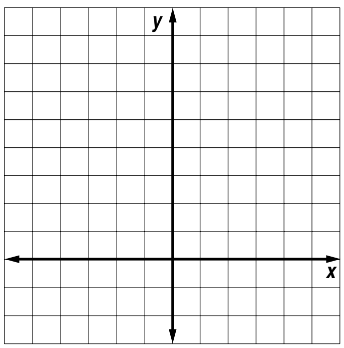
Exercises

1. Sketch the graph of $y = x^2 - 5x + 6$. List the characteristics of functions displayed by this graph.

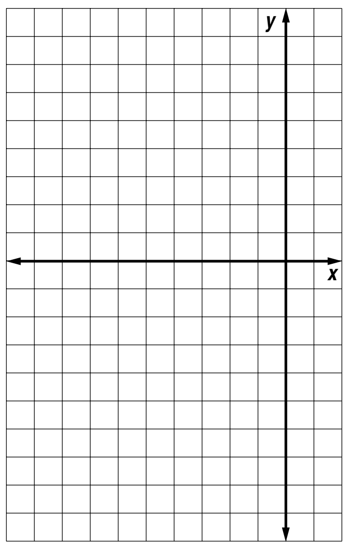


2. What characteristics do absolute value functions and quadratic functions have in common? How do they differ?

3. Graph $y = |x + 3|$.



4. Graph $y = x^2 + 8x + 7$.



Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

8-6 Enrichment

Oblique Asymptotes

The graph of $y = ax + b$, where $a \neq 0$, is called an oblique asymptote of $y = f(x)$ if the graph of f comes closer and closer to the line as $x \to \infty$ or $x \to -\infty$. ∞ is the mathematical symbol for **infinity**, which means *endless*.

For $f(x) = 3x + 4 + \frac{2}{x}$, $y = 3x + 4$ is an oblique asymptote because $f(x) - 3x - 4 = \frac{2}{x}$, and $\frac{2}{x} \to 0$ as $x \to \infty$ or $-\infty$. In other words, as $|x|$ increases, the value of $\frac{2}{x}$ gets smaller and smaller approaching 0.

Example **Find the oblique asymptote for $f(x) = \frac{x^2 + 8x + 15}{x + 2}$.**

$$\begin{array}{r|rrr} -2 & 1 & 8 & 15 \\ & & -2 & -12 \\ \hline & 1 & 6 & 3 \end{array}$$

Use synthetic division.

$$y = \frac{x^2 + 8x + 15}{x + 2} = x + 6 + \frac{3}{x + 2}$$

As $|x|$ increases, the value of $\frac{3}{x + 2}$ gets smaller. In other words, since $\frac{3}{x + 2} \to 0$ as $x \to \infty$ or $x \to -\infty$, $y = x + 6$ is an oblique asymptote.

Use synthetic division to find the oblique asymptote for each function.

1. $y = \frac{8x^2 - 4x + 11}{x + 5}$

2. $y = \frac{x^2 + 3x - 15}{x - 2}$

3. $y = \frac{x^2 - 2x - 18}{x - 3}$

4. $y = \frac{ax^2 + bx + c}{x - d}$

5. $y = \frac{ax^2 + bx + c}{x + d}$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

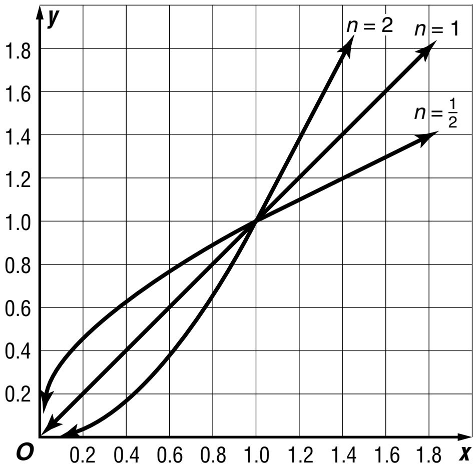
NAME ____ DATE______________ PERIOD _____

9-1 Enrichment

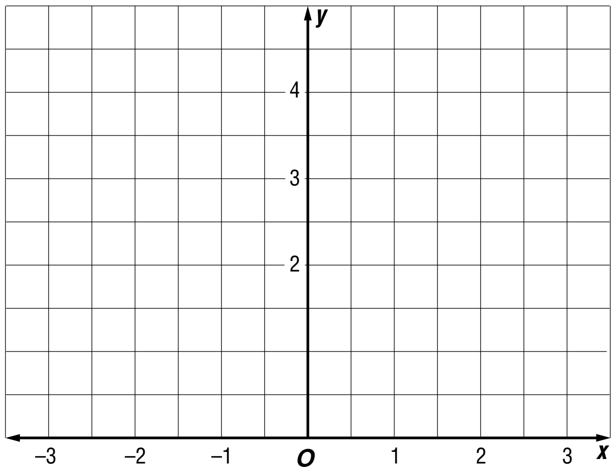
Families of Curves

Use these graphs for the problems below.

The Family $y = x^n$



The Family $y = e^{mx}$



1. Use the graph on the left to describe the relationship among the curves $y = x^{\frac{1}{2}}$, $y = x^1$, and $y = x^2$.

2. Graph $y = x^n$ for $n = \frac{1}{10}, \frac{1}{4}$, 4, and 10 on the grid with $y = x^{\frac{1}{2}}$, $y = x^1$, and $y = x^2$.

3. Which two regions in the first quadrant contain no points of the graphs of the family for $y = x^n$?

4. On the right grid, graph the members of the family $y = e^{mx}$ for which $m = 1$ and $m = -1$.

5. Describe the relationship among these two curves and the y-axis.

6. Graph $y = e^{mx}$ for $m = 0, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm 2$, and ± 4.

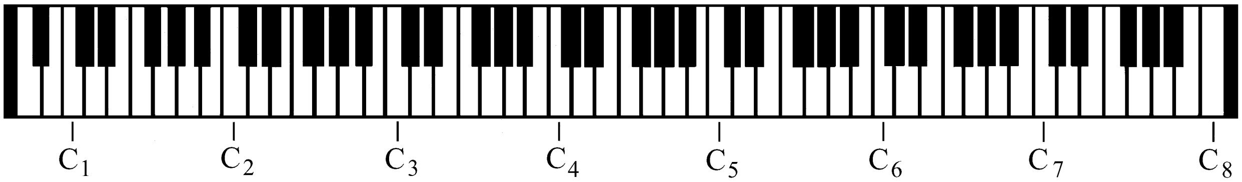
Lesson 9-1

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

9-2 Enrichment

Musical Relationships

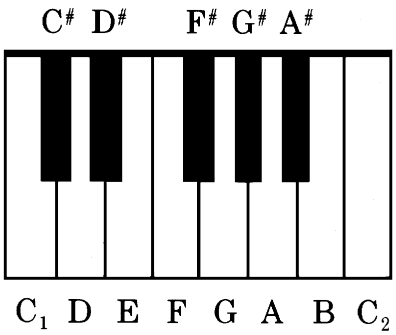
The frequencies of notes that are one octave apart in a musical scale are related by an exponential equation. For the eight C notes on a piano, the equation is $C_n = C_1 2^{n-1}$, where C_n represents the frequency of note C_n.



1. Find the relationship between C_1 and C_2.

2. Find the relationship between C_1 and C_4.

The frequencies of consecutive notes are related by a common ratio r. The general equation is $f_n = f_1 r^{n-1}$.

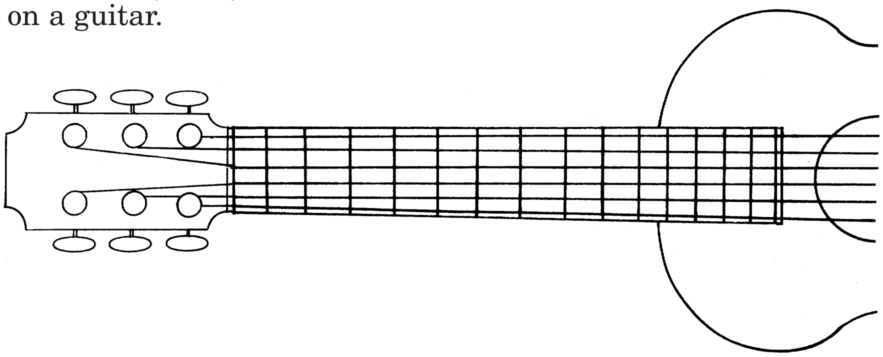


3. If the frequency of middle C is 261.6 cycles per second and the frequency of the next higher C is 523.2 cycles per second, find the common ratio r. (*Hint:* The two C's are 12 notes apart.) Write the answer as a radical expression.

4. Substitute decimal values for r and f_1 to find a specific equation for f_n.

5. Find the frequency of F# above middle C.

6. Frets are a series of ridges placed across the fingerboard of a guitar. They are spaced so that the sound made by pressing a string against one fret has about 1.0595 times the wavelength of the sound made by using the next fret. The general equation is $w_n = w_0(1.0595)^n$. Describe the arrangement of the frets on a guitar.

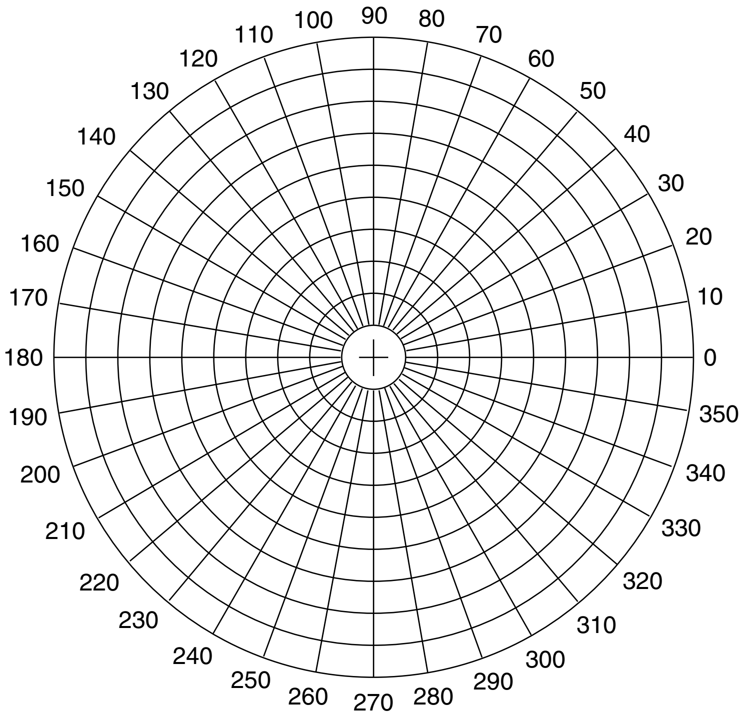


Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

9-3 Enrichment

Spirals

Consider an angle in standard position with its vertex at a point *O* called the pole. Its initial side is on a coordinatized axis called the *polar axis.* A point *P* on the terminal side of the angle is named by the *polar coordinates* (r, θ), where *r* is the directed distance of the point from *O* and θ is the measure of the angle. Graphs in this system may be drawn on polar coordinate paper such as the kind shown below.



1. Use a calculator to complete the table for $\log_2 r = \frac{\theta}{120}$.

(*Hint:* To find θ on a calculator, press 120 [×] [LOG] *r* [)] [÷] [LOG] 2 [)].)

r	1	2	3	4	5	6	7	8

2. Plot the points found in Exercise 1 on the grid above and connect to form a smooth curve.

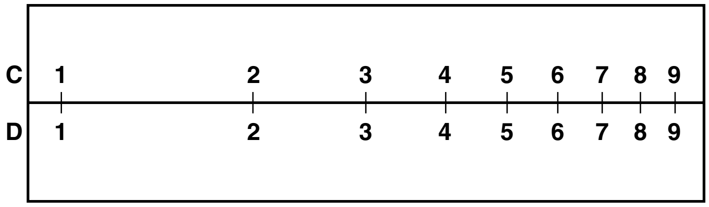
This type of spiral is called a logarithmic spiral because the angle measures are proportional to the logarithms of the radii.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

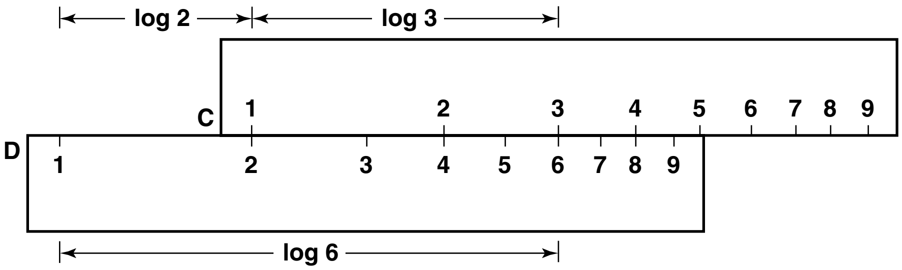
9-4 Enrichment

The Slide Rule

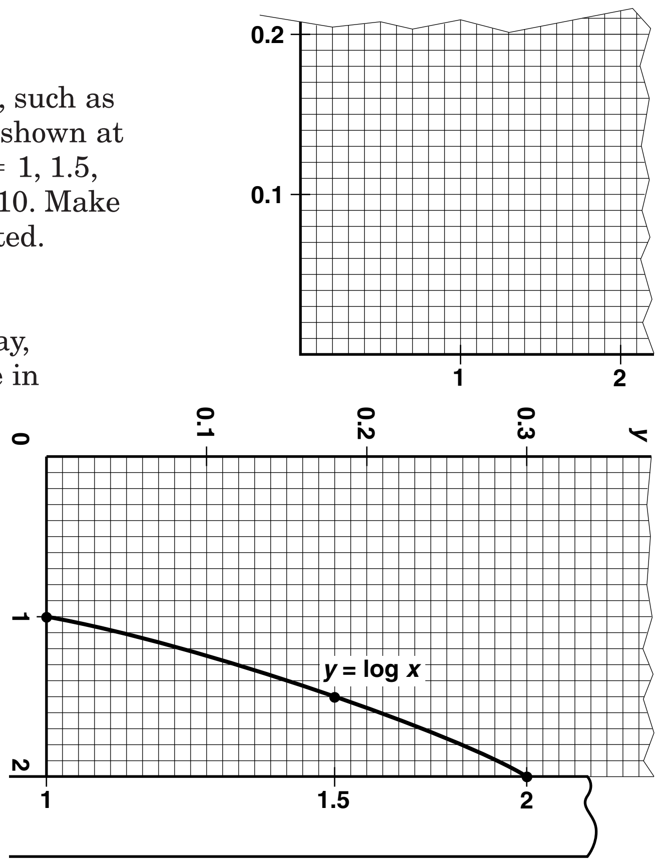
Before the invention of electronic calculators, computations were often performed on a slide rule. A slide rule is based on the idea of logarithms. It has two movable rods labeled with C and D scales. Each of the scales is logarithmic.



To multiply 2 × 3 on a slide rule, move the C rod to the right as shown below. You can find 2 × 3 by adding log 2 to log 3, and the slide rule adds the lengths for you. The distance you get is 0.778, or the logarithm of 6.



Follow the steps to make a slide rule.



1. Use graph paper that has small squares, such as 10 squares to the inch. Using the scales shown at the right, plot the curve $y = \log x$ for $x = 1, 1.5,$ and the whole numbers from 2 through 10. Make an obvious heavy dot for each point plotted.

2. You will need two strips of cardboard. A 5-by-7 index card, cut in half the long way, will work fine. Turn the graph you made in Exercise 1 sideways and use it to mark a logarithmic scale on each of the two strips. The figure shows the mark for 2 being drawn.

3. Explain how to use a slide rule to divide 8 by 2.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ______________________ DATE ____________ PERIOD _____

9-5 Enrichment

Approximations for π and e

The following expression can be used to approximate e. If greater and greater values of n are used, the value of the expression approximates e more and more closely.

$$\left(1 + \frac{1}{n}\right)^n$$

Another way to approximate e is to use this infinite sum. The greater the value of n, the closer the approximation.

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \ldots + \frac{1}{2 \cdot 3 \cdot 4 \cdot \ldots \cdot n} + \ldots$$

In a similar manner, π can be approximated using an infinite product discovered by the English mathematician John Wallis (1616–1703).

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \ldots \cdot \frac{2n}{2n - 1} \cdot \frac{2n}{2n + 1} \ldots$$

Solve each problem.

1. Use a calculator with an e^x key to find e to 7 decimal places.

2. Use the expression $\left(1 + \frac{1}{n}\right)^n$ to approximate e to 3 decimal places. Use 5, 100, 500, and 7000 as values of n.

3. Use the infinite sum to approximate e to 3 decimal places. Use the whole numbers from 3 through 6 as values of n.

4. Which approximation method approaches the value of e more quickly?

5. Use a calculator with a π key to find π to 7 decimal places.

6. Use the infinite product to approximate π to 3 decimal places. Use the whole numbers from 3 through 6 as values of n.

7. Does the infinite product give good approximations for π quickly?

8. Show that $\pi^4 + \pi^5$ is equal to e^6 to 4 decimal places.

9. Which is larger, e^π or π^e?

10. The expression x reaches a maximum value at $x = e$. Use this fact to prove the inequality you found in Exercise 9.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

9-6 Enrichment

Effective Annual Yield

When interest is compounded more than once per year, the effective annual yield is higher than the annual interest rate. The effective annual yield, E, is the interest rate that would give the same amount of interest if the interest were compounded once per year. If P dollars are invested for one year, the value of the investment at the end of the year is $A = P(1 + E)$. If P dollars are invested for one year at a nominal rate r compounded n times per year, the value of the investment at the end of the year is $A = P\left(1 + \frac{r}{n}\right)^n$. Setting the amounts equal and solving for E will produce a formula for the effective annual yield.

$$P(1 + E) = P\left(1 + \frac{r}{n}\right)^n$$

$$1 + E = \left(1 + \frac{r}{n}\right)^n$$

$$E = \left(1 + \frac{r}{n}\right)^n - 1$$

If compounding is continuous, the value of the investment at the end of one year is $A = Pe^r$. Again set the amounts equal and solve for E. A formula for the effective annual yield under continuous compounding is obtained.

$$P(1 + E) = Pe^r$$

$$1 + E = e^r$$

$$E = e^r - 1$$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

Example 1 **Find the effective annual yield of an investment made at 7.5% compounded monthly.**

$r = 0.075$

$n = 12$

$E = \left(1 + \frac{0.075}{12}\right)^{12} - 1 \approx 7.76\%$

Example 2 **Find the effective annual yield of an investment made at 6.25% compounded continuously.**

$r = 0.0625$

$E = e^{0.0625} - 1 \approx 6.45\%$

Exercises

Find the effective annual yield for each investment.

1. 10% compounded quarterly

2. 8.5% compounded monthly

3. 9.25% compounded continuously

4. 7.75% compounded continuously

5. 6.5% compounded daily (assume a 365-day year)

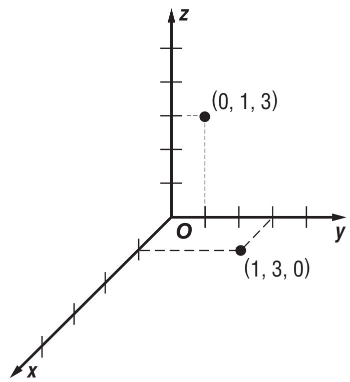
6. Which investment yields more interest—9% compounded continuously or 9.2% compounded quarterly?

10-1 Enrichment

Distance Between Points in Space

The Distance Formula and Midpoint Formula on the coordinate plane is derived from the Pythagorean Theorem $a^2 + b^2 = c^2$.

In three dimensions, the coordinate grid contains the x-axis and the y-axis, as in two-dimensional geometry, and also a z-axis. An example of a line segment drawn on a three-dimensional coordinate grid is shown at the right.



The three-dimensional distance formula is much like the one for two dimensions. The distance from $A\ (x_1, y_1, z_1)$ to $B\ (x_2, y_2, z_2)$ can be found using $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.

Example **Find the distance between the points (1, 3, 0) and (0, 1, 3).**

$d = \sqrt{(1 - 0)^2 + (3 - 1)^2 + (0 - 3)^2}$ Replace (x_1, y_1, z_1) with (1, 3, 0) and (x_2, y_2, z_2) with (0, 1, 3).

$d = \sqrt{1 + 4 + 9}$ or $\sqrt{14}$ Simplify.

Exercises

1. Find the distance between each pair of points A (1, 3, −2) and B (4, 2, 1).

2. Find the distance between each pair of points C (5, 3, 2) and D (0, 1, 7).

3. Find the distance between each pair of points E (−2, −1, 6) and F (−1, 3, 2).

4. Use what you know about the midpoint formula for a segment graphed on a regular coordinate grid to make a conjecture about the formula for finding the coordinates of a midpoint in three-dimensions.

5. Find the midpoint for each segment in Exercises 1–3.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ____ DATE______________ PERIOD _____

10-2 Enrichment

Limits

Sequences of numbers with a rational expression for the general term often approach some number as a finite limit. For example, the reciprocals of the positive integers approach 0 as n gets larger and larger. This is written using the notation shown below. The symbol ∞ stands for infinity and $n \to \infty$ means that n is getting larger and larger, or "n goes to infinity."

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots \qquad \lim_{n \to \infty} \frac{1}{n} = 0$

Example **Find** $\lim_{n \to \infty} \frac{n^2}{(n+1)^2}$

It is not immediately apparent whether the sequence approaches a limit or not. But notice what happens if we divide the numerator and denominator of the general term by n^2.

$$\frac{n^2}{(n+1)^2} = \frac{n^2}{n^2 + 2n + 1}$$

$$= \frac{\frac{n^2}{n^2}}{\frac{n^2}{n^2} + \frac{2n}{n^2} + \frac{1}{n^2}}$$

$$= \frac{1}{1 + \frac{2}{n} + \frac{1}{n^2}}$$

The two fractions in the denominator will approach a limit of 0 as n gets very large, so the entire expression approaches a limit of 1.

Exercises

Find the following limits.

1. $\lim_{n \to \infty} \frac{n^3 + 5n}{n^4 - 6}$

2. $\lim_{n \to \infty} \frac{1 - n}{n^2}$

3. $\lim_{n \to \infty} \frac{2(n+1) + 1}{2n + 1}$

4. $\lim_{n \to \infty} \frac{2n + 1}{1 - 3n}$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

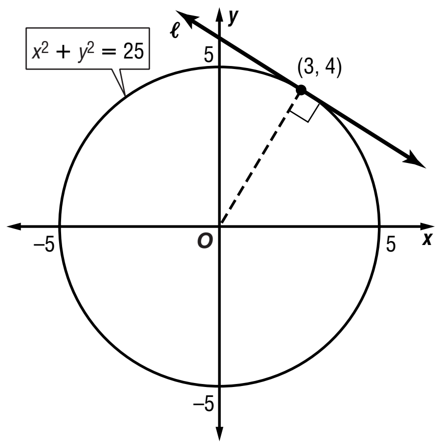
NAME ______________________________ DATE______________ PERIOD _____

10-3 Enrichment

Tangents to Circles

A line that intersects a circle in exactly one point is a **tangent** to the circle. In the diagram, line ℓ is tangent to the circle with equation $x^2 + y^2 = 25$ at the point whose coordinates are (3, 4).

A line is tangent to a circle at a point P on the circle if and only if the line is perpendicular to the radius from the center of the circle to point P. This fact enables you to find an equation of the tangent to a circle at a point P if you know an equation for the circle and the coordinates of P.



Exercises

Use the diagram above to solve each problem.

1. What is the slope of the radius to the point with coordinates (3, 4)? What is the slope of the tangent to that point?

2. Find an equation of the line ℓ that is tangent to the circle at (3, 4).

3. If k is a real number between -5 and 5, how many points on the circle have x-coordinate k? State the coordinates of these points in terms of k.

4. Describe how you can find equations for the tangents to the points you named for Exercise 3.

5. Find an equation for the tangent at $(-3, 4)$.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

10-4 Enrichment

Eccentricity

In an ellipse, the ratio $\frac{c}{a}$ is called the **eccentricity** and is denoted by the letter e. Eccentricity measures the elongation of an ellipse. The closer e is to 0, the more an ellipse looks like a circle. The closer e is to 1, the more elongated it is. Recall that the equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ where a is the length of the major axis, and that $c = \sqrt{a^2 - b^2}$.

Find the eccentricity of each ellipse rounded to the nearest hundredth.

1. $\frac{x^2}{9} + \frac{y^2}{36} = 1$

2. $\frac{x^2}{81} + \frac{y^2}{9} = 1$

3. $\frac{x^2}{4} + \frac{y^2}{9} = 1$

4. $\frac{x^2}{16} + \frac{y^2}{9} = 1$

5. $\frac{x^2}{36} + \frac{y^2}{16} = 1$

6. $\frac{x^2}{4} + \frac{y^2}{36} = 1$

7. Is a circle an ellipse? Explain your reasoning.

8. The center of the sun is one focus of Earth's orbit around the sun. The length of the major axis is 186,000,000 miles, and the foci are 3,200,000 miles apart. Find the eccentricity of Earth's orbit.

9. An artificial satellite orbiting the earth travels at an altitude that varies between 132 miles and 583 miles above the surface of the earth. If the center of the earth is one focus of its elliptical orbit and the radius of the earth is 3950 miles, what is the eccentricity of the orbit?

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ____ DATE______________ PERIOD _____

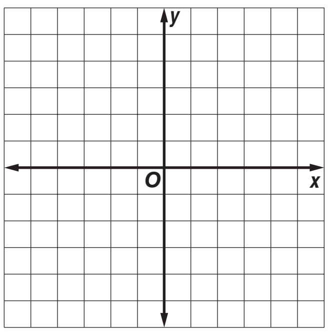
10-5 Enrichment

Rectangular Hyperbolas

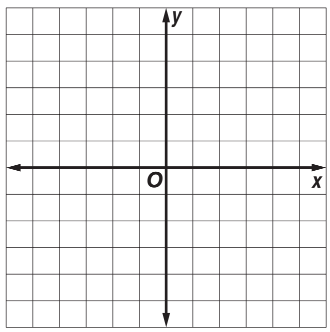
A **rectangular hyperbola** is a hyperbola with perpendicular asymptotes. For example, the graph of $x^2 - y^2 = 1$ is a rectangular hyperbola. A hyperbola with asymptotes that are not perpendicular is called a **nonrectangular hyperbola**. The graphs of equations of the form $xy = c$, where c is a constant, are rectangular hyperbolas.

Make a table of values and plot points to graph each rectangular hyperbola below. Be sure to consider negative values for the variables.

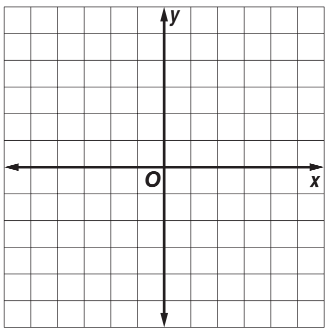
1. $xy = -4$



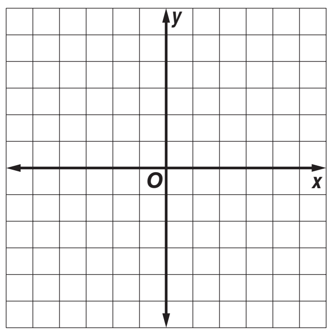
2. $xy = 3$



3. $xy = -1$



4. $xy = 8$



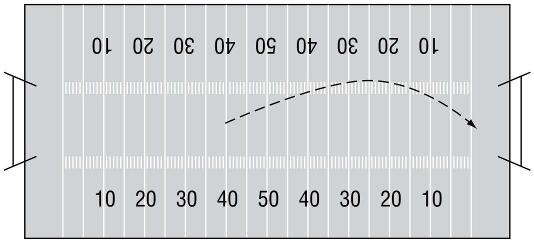
5. Make a conjecture about the asymptotes of rectangular hyperbolas.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

10-6 Enrichment

Parabolic Football

A parabola is defined as all the points (x, y) in the plane whose distance from a fixed point, called the **focus** is the same as its distance from a fixed line, called the **directrix**. Examples of parabolas are the cables on a suspension bridge, satellite dishes, and the flight path of a football during a kick-off.



At the kick-off at the beginning of a football game the ball is placed on the 40-yard line of the kicking team. Suppose the receiving team catches the ball on the goal line. Assume the 50-yard line has coordinates (0, 0) and the 40-yard line of the kicking team has coordinates (−10, 0).

1. Determine which equation of the parabola best describes this situation. For the other choices explain why they do not make sense to the situation.

a. $y = 50x(x + 10)$

b. $y = 100(x - 50)(x + 10)$

c. $y = -\frac{1}{4}(x - 50)(x + 10)$

d. $y = -20(x - 50)(x - 60)$

2. During the same game, the quarterback throws a forward pass from the 50-yard line to his receiver on the 25-yard line. Assuming the ball follows the path of a parabola, write an equation modeling the flight path of the ball from quarterback to receiver.

3. The team did not pick up the first down, so they elect to try a field goal. Fortunately, one of the assistant coaches is a part-time mathematician and found an equation that describes their kicker as: $y = -0.2x^2 + 8x$. The line of scrimmage is the 25-yard line, so the ball will be placed on the 32-yard line for the kick. Add 10 more yards for the depth of the end zone (goal line to the goal post), making it a 42-yard field goal attempt. Will your kicker make it?

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

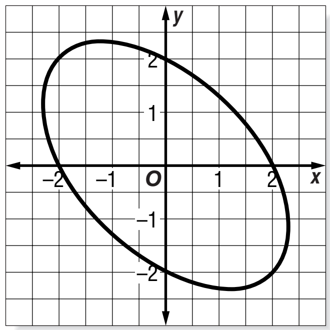
NAME ______________________________ DATE______________ PERIOD _____

10-7 Enrichment

Graphing Quadratic Equations with xy-Terms

You can use a graphing calculator to examine graphs of quadratic equations that contain *xy*-terms.

Example **Use a graphing calculator to display the graph of $x^2 + xy + y^2 = 4$.**



Solve the equation for y in terms of x by using the quadratic formula.

$y^2 + xy + (x^2 - 4) = 0$

To use the formula, let $a = 1$, $b = x$, and $c = (x^2 - 4)$.

$$y = \frac{-x \pm \sqrt{x^2 - 4(1)(x^2 - 4)}}{2}$$

$$y = \frac{-x \pm \sqrt{16 - 3x^2}}{2}$$

To graph the equation on the graphing calculator, enter the two equations:

$$y = \frac{-x + \sqrt{16 - 3x^2}}{2} \text{ and } y = \frac{-x - \sqrt{16 - 3x^2}}{2}$$

Exercises

Use a graphing calculator to graph each equation. State the type of curve each graph represents.

1. $y^2 + xy = 8$

2. $x^2 + y^2 - 2xy - x = 0$

3. $x^2 - xy + y^2 = 15$

4. $x^2 + xy + y^2 = -9$

5. $2x^2 - 2xy - y^2 + 4x = 20$

6. $x^2 - xy - 2y^2 + 2x + 5y - 3 = 0$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

11-1 Enrichment

Fibonacci Sequence

Leonardo Fibonacci first discovered the sequence of numbers named for him while studying rabbits. He wanted to know how many pairs of rabbits would be produced in n months, starting with a single pair of newborn rabbits. He made the following assumptions.

1. Newborn rabbits become adults in one month.
2. Each pair of rabbits produces one pair each month.
3. No rabbits die.

Let F_n represent the number of pairs of rabbits at the end of n months. If you begin with one pair of newborn rabbits, $F_0 = F_1 = 1$. This pair of rabbits would produce one pair at the end of the second month, so $F_2 = 1 + 1$, or 2. At the end of the third month, the first pair of rabbits would produce another pair. Thus, $F_3 = 2 + 1$, or 3.

The chart below shows the number of rabbits each month for several months.

Month	Adult Pairs	Newborn Pairs	Total
F_0	0	1	1
F_1	1	0	1
F_2	1	1	2
F_3	2	1	3
F_4	3	2	5
F_5	5	3	8

Exercises

Solve.

1. Starting with a single pair of newborn rabbits, how many pairs of rabbits would there be at the end of 12 months?

2. Write the first 10 terms of the sequence for which $F_0 = 3$, $F_1 = 4$, and $F_n = F_{n-2} + F_{n-1}$.

3. Write the first 10 terms of the sequence for which $F_0 = 1$, $F_1 = 5$, $F_n = F_{n-2} + F_{n-1}$.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

11-2 Enrichment

Arithmetic Series in Computer Programming

Arithmetic series are used in the analysis of the efficiency of computer programs. Computers effortlessly automate time consuming, often repetitive tasks such as addition and multiplication of numbers. These repetitive tasks are carried out using a *Loop* statement provided by a programming language to execute the calculations until a logical condition is, or is not, satisfied. The loop usually repeats a calculation followed by an assignment statement, which is assigning the number to a specific memory location in the computer.

Suppose you were writing a program to calculate the sum of the numbers from 1 to 10, that is 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10. Two algorithms to calculate this series are shown in the table with the sequential step of the algorithm in the left column.

Step Number	Algorithm
1	Assign in memory s = 1
2	Assign j = 2
3	If j < 11 then do steps 4 and 5
4	Assign s = s + j
5	Assign j = j + 1

1. Write an algorithm segment in pseudo-code (like in the table) which for any given values of a, d, and n—the initial value, the common difference, and the number of terms in the progression, respectively—computes the sum of the series, $\sum_{i=0}^{n} a + id$.

2. Double summations are used to analyze nested loops (loops inside of loops). Calculate the double sums below. Start with the inner summation first and then proceed to the outer summation.

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i + 2i + 3i) = \sum_{i=1}^{4} 6i = 6 + 12 + 18 + 24 = 60.$$

Also recall the sum of a arithmetic series is equal to $\frac{n}{2}(a_1 + a_n)$, where n is the number of terms in the series, a_1 is the first term of the sequence and a_n is the last term.

a). $\sum_{i=1}^{2}\sum_{j=1}^{2}(2i + 3j)$

b). $\sum_{i=1}^{n}\sum_{j=1}^{m} ij$

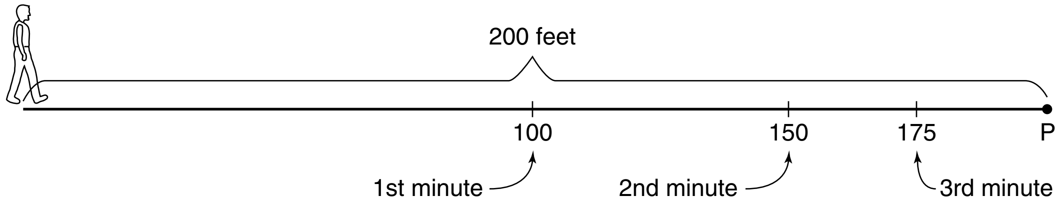
Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ____ DATE______________ PERIOD _____

11-3 Enrichment

Half the Distance

Suppose you are 200 feet from a fixed point, *P*. Suppose that you are able to move to the halfway point in one minute, to the next halfway point one minute after that, and so on.



An interesting sequence results because according to the problem, you never actually reach the point *P*, although you do get arbitrarily close to it.

You can compute how long it will take to get within some specified small distance of the point. On a calculator, you enter the distance to be covered and then count the number of successive divisions by 2 necessary to get within the desired distance.

Example **How many minutes are needed to get within 0.1 foot of a point 200 feet away?**

Count the number of times you divide by 2.

Enter: 200 [÷] 2 [ENTER] [÷] 2 [ENTER] [÷] 2 [ENTER], and so on

Result: 0.0976562

You divided by 2 eleven times. The time needed is 11 minutes.

Exercises **Use the method illustrated above to solve each problem.**

1. If it is about 2500 miles from Los Angeles to New York, how many minutes would it take to get within 0.1 mile of New York? How far from New York are you at that time?

2. If it is 25,000 miles around Earth, how many minutes would it take to get within 0.5 mile of the full distance around Earth? How far short would you be?

3. If it is about 250,000 miles from Earth to the Moon, how many minutes would it take to get within 0.5 mile of the Moon? How far from the surface of the Moon would you be?

4. If it is about 30,000,000 feet from Honolulu to Miami, how many minutes would it take to get to within 1 foot of Miami? How far from Miami would you be at that time?

5. If it is about 93,000,000 miles to the sun, how many minutes would it take to get within 500 miles of the sun? How far from the sun would you be at that time?

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

11-4 Enrichment

Annuities

An annuity is a fixed amount of money payable at given intervals. For example, suppose you wanted to set up a trust fund so that $30,000 could be withdrawn each year for 14 years before the money ran out. Assume the money can be invested at 9%.

You must find the amount of money that needs to be invested. Call this amount A. After the third payment, the amount left is

$1.09[1.09A - 30{,}000(1 + 1.09)] - 30{,}000 = 1.09^2A - 30{,}000(1 + 1.09 + 1.09^2)$.

The results are summarized in the table below.

Payment Number	Number of Dollars Left After Payment
1	$A - 30{,}000$
2	$1.09A - 30{,}000(1 + 1.09)$
3	$1.09^2A - 30{,}000(1 + 1.09 + 1.09^2)$

1. Use the pattern shown in the table to find the number of dollars left after the fourth payment.

2. Find the amount left after the tenth payment.

The amount left after the 14th payment is $1.09^{13}A - 30{,}000(1 + 1.09 + 1.09^2 + \ldots + 1.09^{13})$. However, there should be no money left after the 14th and final payment.

$1.09^{13}A - 30{,}000(1 + 1.09 + 1.09^2 + \ldots + 1.09^{13}) = 0$

Notice that $1 + 1.09 + 1.09^2 + \ldots + 1.09^{13}$ is a geometric series where $a_1 = 1$, $a_n = 1.09^{13}$, $n = 14$ and $r = 1.09$.

Using the formula for S_n,

$$1 + 1.09 + 1.09^2 + \ldots + 1.09^{13} = \frac{a_1 - a_1r^n}{1 - r} = \frac{1 - 1.09^{14}}{1 - 1.09} = \frac{1 - 1.09^{14}}{-0.09}.$$

3. Show that when you solve for A you get $A = \frac{30{,}000}{0.09}\left(\frac{1.09^{14} - 1}{1.09^{13}}\right)$.

Therefore, to provide $30,000 for 14 years where the annual interest rate is 9%, you need $\frac{30{,}000}{0.09}\left(\frac{1.09^{14} - 1}{1.09^{13}}\right)$ dollars.

4. Use a calculator to find the value of A in problem 3.

In general, if you wish to provide P dollars for each of n years at an annual rate of r%, you need A dollars where

$$\left(1 + \frac{r}{100}\right)^{n-1} A - P\left[1 + \left(1 + \frac{r}{100}\right) + \left(1 + \frac{r}{100}\right)^2 + \ldots + \left(1 + \frac{r}{100}\right)^{n-1}\right] = 0.$$

You can solve this equation for A, given P, n, and r.

Lesson 11-4

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ____ DATE______________ PERIOD _____

11-5 Enrichment

Infinite Continued Fractions

Some infinite expressions are actually equal to real numbers! The infinite continued fraction at the right is one example.

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}}$$

If you use x to stand for the infinite fraction, then the entire denominator of the first fraction on the right is also equal to x. This observation leads to the following equation:

$$x = 1 + \frac{1}{x}$$

Write a decimal for each continued fraction.

1. $1 + \frac{1}{1}$

2. $1 + \cfrac{1}{1 + \frac{1}{1}}$

3. $1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1}}}$

4. $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1}}}}$

5. $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1}}}}}$

6. The more terms you add to the fractions above, the closer their value approaches the value of the infinite continued fraction. What value do the fractions seem to be approaching?

7. Rewrite $x = 1 + \frac{1}{x}$ as a quadratic equation and solve for x.

8. Find the value of the following infinite continued fraction.

$$3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \dots}}}}$$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ____ DATE______________ PERIOD _____

11-6 Enrichment

Lesson 11-6

Continued Fractions

The fraction below is an example of a continued fraction. Note that each fraction in the continued fraction has a numerator of 1.

$$2 + \cfrac{1}{3 + \cfrac{1}{4 + \frac{1}{5}}}$$

Example 1 **Evaluate the continued fraction above. Start at the bottom and work your way up.**

Step 1: $4 + \frac{1}{5} = \frac{20}{5} + \frac{1}{5} = \frac{21}{5}$

Step 2: $\frac{1}{\frac{21}{5}} = \frac{5}{21}$

Step 3: $3 + \frac{5}{21} = \frac{63}{21} + \frac{5}{21} = \frac{68}{21}$

Step 4: $\frac{1}{\frac{68}{21}} = \frac{21}{68}$

Step 5: $2 + \frac{21}{68} = 2\frac{21}{68}$

Example 2 **Change $\frac{25}{11}$ into a continued fraction.**

Follow the steps.

Step 1: $\frac{25}{11} = \frac{22}{11} + \frac{3}{11} = 2 + \frac{3}{11}$

Step 2: $\frac{3}{11} = \frac{1}{\frac{11}{3}}$

Step 3: $\frac{11}{3} = \frac{9}{3} + \frac{2}{3} = 3 + \frac{2}{3}$

Step 4: $\frac{2}{3} = \frac{1}{\frac{3}{2}}$

Step 5: $\frac{3}{2} = \frac{2}{2} + \frac{1}{2} = 1 + \frac{1}{2}$

Stop, because the numerator is 1.

Thus, $\frac{25}{11}$ can be written as $2 + \cfrac{1}{3 + \cfrac{1}{1 + \frac{1}{2}}}$

Evaluate each continued fraction.

1. $1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \frac{1}{3}}}}$

2. $0 + \cfrac{1}{6 + \cfrac{1}{4 + \frac{1}{2}}}$

3. $2 + \cfrac{1}{4 + \cfrac{1}{6 + \cfrac{1}{8 + \frac{1}{10}}}}$

4. $5 + \cfrac{1}{7 + \cfrac{1}{9 + \frac{1}{11}}}$

Change each fraction into a continued fraction.

5. $\frac{75}{31}$

6. $\frac{29}{8}$

7. $\frac{13}{19}$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

11-7 Enrichment

Patterns in Pascal's Triangle

You have learned that the coefficients in the expansion of $(x + y)^n$ yield a number pyramid called **Pascal's triangle.**

Row 1 →							1							
Row 2 →						1		1						
Row 3 →					1		2		1					
Row 4 →				1		3		3		1				
Row 5 →			1		4		6		4		1			
Row 6 →		1		5		10		10		5		1		
Row 7 →	1		6		15		20		15		6		1	

As many rows can be added to the bottom of the pyramid as you please.

This activity explores some of the interesting properties of this famous number pyramid.

1. Pick a row of Pascal's triangle.

a. What is the sum of all the numbers in all the rows *above* the row you picked?

b. What is the sum of all the numbers in the row you picked?

c. How are your answers for parts **a** and **b** related?

d. Repeat parts **a** through **c** for at least three more rows of Pascal's triangle. What generalization seems to be true?

e. See if you can prove your generalization.

2. Pick any row of Pascal's triangle that comes after the first.

a. Starting at the left end of the row, add the first number, the third number, the fifth number, and so on. State the sum.

b. In the same row, add the second number, the fourth number, and so on. State the sum.

c. How do the sums in parts **a** and **b** compare?

d. Repeat parts **a** through **c** for at least three other rows of Pascal's triangle. What generalization seems to be true?

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

11-8 Enrichment

Proof by Induction

Mathematical induction is a useful tool when you want to prove that a statement is true for all natural numbers.

The three steps in using induction are:

1. Prove that the statement is true for $n = 1$.

2. Prove that if the statement is true for the natural number n, it must also be true for $n + 1$.

3. Conclude that the statement is true for all natural numbers.

Follow the steps to complete each proof.

Theorem A: The sum of the first n odd natural numbers is equal to n^2.

1. Show that the theorem is true for $n = 1$.

2. Suppose $1 + 3 + 5 + \ldots + (2n - 1) = n^2$. Show that $1 + 3 + 5 + \ldots + (2n - 1) + (2n + 1) = (n + 1)^2$.

3. Summarize the results of problems 1 and 2.

Theorem B: Show that $a^n - b^n$ is exactly divisible by $a - b$ for n equal to 1, 2, 3, and all natural numbers.

4. Show that the theorem is true for $n = 1$.

5. The expression $a^{n+1} - b^{n+1}$ can be rewritten as $a(a^n - b^n) + b^n(a - b)$. Verify that this is true.

6. Suppose $a - b$ is a factor of $a^n - b^n$. Use the result in problem 5 to show that $a - b$ must then also be a factor of $a^{n+1} - b^{n+1}$.

7. Summarize the results of problems 4 through 6.

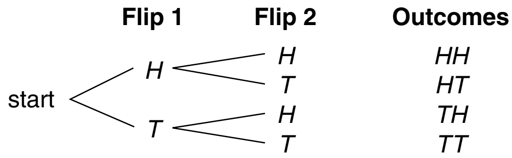
Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ______________________________ DATE ____________ PERIOD _____

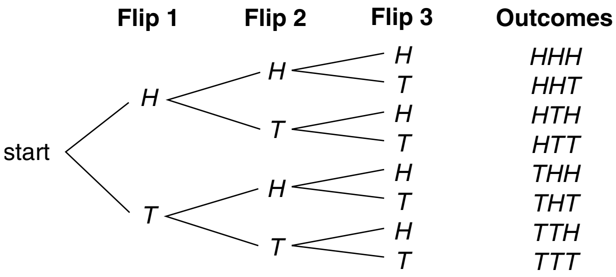
12-1 Enrichment

Tree Diagrams and the Power Rule

If you flip a coin once, there are two possible outcomes: heads showing (H) or tails showing (T). The tree diagram to the right shows the four (2^2) possible outcomes if you flip a coin twice.

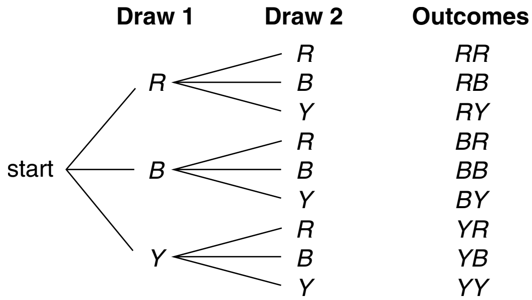


Example 1 **Draw a tree diagram to show all the possible outcomes for flipping a coin three times. List the outcomes.**



There are eight (2^3) possible outcomes. With each extra flip, the number of outcomes doubles. With 4 flips, there would be sixteen (2^4) outcomes.

Example 2 **In a cup there are a red, a blue, and a yellow marble. How many possible outcomes are there if you draw one marble at random, replace it, and then draw another?**



There are nine (3^2) possible outcomes.

The Power Rule for the number of outcomes states that if an experiment is repeated n times, and if there are b possible outcomes each time, there are b^n total possible outcomes.

Find the total number of possible outcomes for each experiment. Use tree diagrams to help you.

1. flipping a coin 5 times

2. doing the marble experiment 6 times

3. flipping a coin 8 times

4. rolling a 6-sided die 2 times

5. rolling a 6-sided die 3 times

6. rolling a 4-sided die 2 times

7. rolling a 4-sided die 3 times

8. rolling a 12-sided die 2 times

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ____ DATE______________ PERIOD _____

12-2 Enrichment

Combinations and Pascal's Triangle

Pascal's triangle is a special array of numbers invented by Blaise Pascal (1623–1662). The values in Pascal's triangle can be found using the combinations shown below.

1. Evaluate the expression in each cell of the triangle.

C(1,0) C(1,1)

C(2,0) C(2,1) C(2,2)

C(3,0) C(3,1) C(3,2) C(3,3)

C(4,0) C(4,1) C(4,2) C(4,3) C(4,4)

C(5,0) C(5,1) C(5,2) C(5,3) C(5,4) C(5,5)

2. The pattern shows the relationship between $C(n, r)$ and Pascal's triangle. In general, it is true that $C(n, r) + C(n, r + 1) = C(n + 1, r + 1)$. Complete the proof of this property. In each step, the denominator has been given.

$$C(n, r) + C(n, r + 1) = \frac{\qquad}{r!(n - r)!} + \frac{\qquad}{(r + 1)!(n - r - 1)!}$$

$$= \frac{\qquad}{r!(n - r)!(r + 1)} + \frac{\qquad}{(r + 1)!(n - r - 1)!(n - r)}$$

$$= \frac{\qquad}{(r + 1)!(n - r)!} + \frac{\qquad}{(r + 1)!(n - r)!}$$

$$= \frac{\qquad}{(r + 1)!(n - r)!}$$

$$= \frac{\qquad}{(r + 1)!(n - r)!}$$

$$= \frac{\qquad}{(r + 1)!(n - r)!}$$

$$= \frac{\qquad}{(r + 1)![(n + 1) - (r + 1)]!}$$

$$= C(n + 1, r + 1)$$

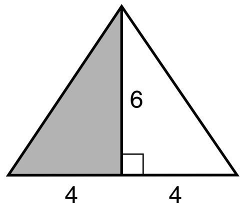
Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME _______________ DATE _______________ PERIOD _____

12-3 Enrichment

Geometric Probability

If a dart, thrown at random, hits the triangular board shown at the right, what is the chance that it will hit the shaded region? This chance, also called a probability, can be determined by comparing the area of the shaded region to the area of the board. This ratio indicates what fraction of the tosses should hit in the shaded region.



$$\frac{\text{area of shaded region}}{\text{area of triangular board}} = \frac{\frac{1}{2}(4)(6)}{\frac{1}{2}(8)(6)}$$

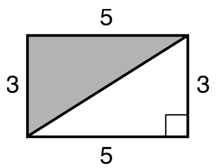
$$= \frac{12}{24} \text{ or } \frac{1}{2}$$

In general, if S is a subregion of some region R, then the probability, $P(S)$, that a point, chosen at random, belongs to subregion S is given by the following.

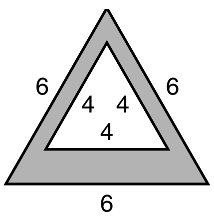
$$P(S) = \frac{\text{area of subregion } S}{\text{are of region } R}$$

Find the probability that a point, chosen at random, belongs to the shaded subregions of the following regions.

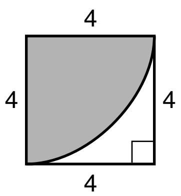
1.



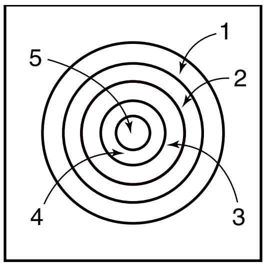
2.



3.



The dart board shown at the right has 5 concentric circles whose centers are also the center of the square board. Each side of the board is 38 cm, and the radii of the circles are 2 cm, 5 cm, 8 cm, 11 cm, and 14 cm. A dart hitting within one of the circular regions scores the number of points indicated on the board, while a hit anywhere else scores 0 points. If a dart, thrown at random, hits the board, find the probability of scoring the indicated number of points.



4. 0 points

5. 1 point

6. 2 points

7. 3 points

8. 4 points

9. 5 points

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

12-4 Enrichment

Conditional Probability

Suppose a pair of dice is thrown. It is known that the sum is greater than seven. Find the probability that the dice match.

The probability of an event given the occurrence of another event is called *conditional probability.* The conditional probability of event A, the dice match, given event B, their sum is greater than seven, is denoted $P(A/B)$.

There are 15 sums greater than seven and there are 36 possible pairs altogether.

$P(B) = \frac{15}{36}$

There are three matching pairs greater than seven.

$P(A \text{ and } B) = \frac{3}{36}$

$$P(A/B) = \frac{P(A \text{ and } B)}{P(B)}$$

$$P(A/B) = \frac{\frac{3}{36}}{\frac{15}{36}} \text{ or } \frac{1}{5}$$

The conditional probability is $\frac{1}{5}$.

A card is drawn from a standard deck of 52 and is found to be red. Given that event, find each of the following probabilities.

1. P(heart)

2. P(ace)

3. P(face card)

4. P(jack or ten)

5. P(six of spades)

6. P(six of hearts)

A sports survey taken at Stirers High School shows that 48% of the respondents liked soccer, 66% liked basketball, and 38% liked hockey. Also, 30% liked soccer and basketball, 22% liked basketball and hockey and 28% liked soccer and hockey. Finally, 12% liked all three sports. Find each of the following probabilities.

7. The probability Meg likes soccer if she likes basketball.

8. The probability Biff likes basketball if he likes soccer.

9. The probability Muffy likes hockey if she likes basketball.

10. The probability Greg likes hockey and basketball if he likes soccer.

Lesson 12-4

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ____ DATE______________ PERIOD _____

12-5 Enrichment

Probability and Tic-Tac-Toe

What would be the chances of winning at tic-tac-toe if it were turned into a game of pure chance? To find out, the nine cells of the tic-tac-toe board are numbered from 1 to 9 and nine chips (also numbered from 1 to 9) are put into a bag. Player A draws a chip at random and enters an *X* in the corresponding cell. Player B does the same and enters an *O*.

To solve the problem, assume that both players draw all their chips without looking and all *X* and *O* entries are made at the same time. There are four possible outcomes: a draw, A wins, B wins, and either A or B can win.

There are 16 arrangements that result in a draw. Reflections and rotations must be counted as shown below.

o x o	x o x	o o x
x o x 4	o o x 4	x x o 8
x o x	x x o	o x x

There are 36 arrangements in which either player may win because both players have winning triples.

x x x	x x x	x o x	x x x	x x x	x x o
o o o 4	x o x 4	x x x 4	x x o 8	o o o 8	x x x 8
x o x	o o o	o o o	o o o	x x o	o o o

In these 36 cases, A's chances of winning are $\frac{13}{40}$.

1. Find the 12 arrangements in which B wins and A cannot.

2. Below are 12 of the arrangements in which A wins and B cannot. Write the numbers to show the reflections and rotations for each arrangement. What is the total number?

o x o	x o x	x x x	x x x	x o o	x o o
x x x	o x o	x o o	o x o	x x x	x x o
o x o	x o x	x o o	o x o	o o x	o o x

x x o	x x x	x x x	x x x	x o o	x x o
o x x	o x o	x o o	x o o	x x x	o x o
o o x	o o x	o x o	o o x	o x o	x o x

3. There are $\frac{9!}{(5!4!)}$ different and equally probable distributions. Complete the chart to find the probability for a draw or for A or B to win.

Draw: $\frac{16}{126}$				=	______
A wins:	______	+	$\frac{13}{40}\left(\frac{36}{126}\right)$	=	______
B wins:	______	+	______	=	______

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

12-6 Enrichment

Standard Deviation of Sample Data

A *population* is the set of all measurements of interest to an investigator. A *sample* is a subset of measurements selected from the population of interest. A *statistic* is any quantity whose value can be calculated from sample data. A common mistake is to use the terms *probability* and *statistics* interchangeably. Probabilities are used to make statements from a population to a sample, but statistics are calculated from a sample and are to make inferences about a population.

The *range* is a statistic calculated by taking the difference between the largest observation and the smallest observation. Range = $x_{max} - x_{min}$.

The *sample variance* is calculated using the formula: $s^2 = \frac{\sum_{i=1}^{n}(\overline{x} - x_i)^2}{n - 1}$ where $\overline{x}$ is the sample mean. Therefore, the *sample standard deviation* is the square root of the sample variance, $s = \sqrt{s^2}$.

To calculate the sample variance:

1. Calculate the sample mean. For example, suppose a sample contains the numbers {2, 5, 6, 9, 11}. The sample mean is $\overline{x} = \frac{2 + 5 + 6 + 9 + 11}{5} = 6.6$.

2. Next use the formula above to calculate the sample variance, in this case:

$$s^2 = \frac{(6.6 - 2)^2 + (6.6 - 5)^2 + (6.6 - 6)^2 + (6.6 - 9)^2 + (6.6 - 11)^2}{4} = 12.3.$$

3. Finally, the sample standard deviation is equal to 3.507 by taking the square root of 12.3.

Exercises

1. What are some differences in the formula for the sample variance compared to the formula for the population variance?

2. Given the random sample {5, 7, 1, 2, 4}, find the sample variance.

3. Calculate the sample standard deviation.

4. Calculate the range of the sample data {5, 7, 1, 2, 4}.

5. An approximation for the sample standard deviation is given by: $s \approx \frac{Range}{4}$. Compare this answer to your answer from 3.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

12-7 Enrichment

Calculating Z-Scores

The normal distribution is the most important probability distribution. Many physical measurements have distributions approximately normal. Examples include height, weight, and measures of intelligence. More importantly, even if the individual variables are not normally distributed, sums and averages tend to still be normally distributed. Unfortunately, normal probability distribution functions are difficult to calculate. Fortunately, statisticians have compiled a table for a normal distribution with mean of zero and standard deviation of one. This is called the Standard Normal Distribution and is typically denoted by N(0, 1), where the N indicates a normal distribution which has mean, μ (mu) = 0, and standard deviation, σ (sigma) = 1.

Suppose the variable x is normally distributed with mean μ and standard deviation σ. In order to calculate probabilities of this normal distribution, we must standardize the variable x by an appropriate transformation. The letter Z denotes the transformed variable and is called the Z-score, which is a measure of relative standing. The following steps are needed to complete the transformation.

- If the mean and standard deviation are not given, then calculate the mean and standard deviation of the given (population) data.
- Define $Z = \frac{x - \mu}{\sigma}$.

Example Find the standard normal variable Z given and $\mu = 15$ and $\sigma = 3$.

Apply the transform to the variable X using the definition above, that is: $Z = \frac{X - 15}{3}$.

1. Suppose that the time, X, to complete an exam is normally distributed. The time, in minutes, of a class of 12 to complete the exam is given in the table. Transform X to a Z-score.

Student	1	2	3	4	5	6	7	8	9	10	11	12
Time	35	42	48	33	32	39	40	52	48	34	36	44

2. Suppose that a random variable X is normally distributed with $\mu = 20$ and $\sigma = 5$. Convert the following probability statements to the equivalent statements by standardizing X.

Example $P(X < 25) = P\left(\frac{X - 20}{5} < 25\right) = P(Z < 25)$

a. $P(X > 18)$

b. $P(17 < X < 23)$

c. $P(X < 19)$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ____ DATE______________ PERIOD _____

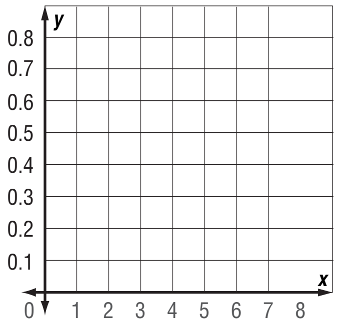
12-8 Enrichment

Exploring Binomial Distribution Functions

In Lesson 12-8, you calculated probabilities using binomial distributions. In this exercise, you will explore graphs of binomial distributions.

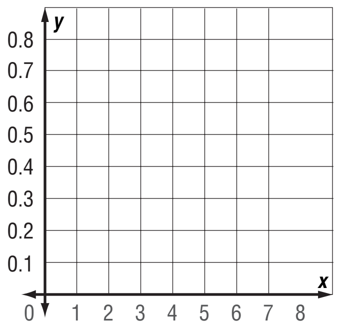
1. Bob is the pitcher on your baseball team. The probability that he will throw a strike is 50%. If Bob throws six pitches, fill in the probabilities for each number of strikes in the table below. Then graph the number of strikes (s) versus probability (p) on the coordinate grid provided.

Strikes (*s*)	Probability (*p*)
0	
1	
2	
3	
4	
5	
6	



2. Joy's pug is going to have puppies. The probability that a puppy will be black is 25%. If there are 6 puppies in the litter, fill in the table below with the probability that corresponds to each possible number of black puppies. Then graph the number of black puppies (b) versus probability(p) on the coordinate grid provided.

Black Puppies (*b*)	Probability (*p*)
0	
1	
2	
3	
4	
5	
6	



3. How do the graphs relate to the distribution curves that you studied in Lesson 12-7?

4. What general shape would you expect the graph of a binomial distribution with a 50% probability of success to look like? With a probability less than 50%? With a probability greater than 50%?

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ____ DATE______________ PERIOD _____

12-9 Enrichment

Multinomial Experiments

A multinomial is a generalization of a binomial. For example, $(a + b + c)^2$ is a multinomial. One way to determine the coefficients is by direct multiplication using the distributive property. Take each term in the first factor and multiply by each term (a, b, and c) in the second factor, then combine like terms. (Notice that the sum of the exponents is always equal to two.)

$$\begin{aligned}(a + b + c)^2 = (a + b + c)(a + b + c) &= a(a + b + c) + b(a + b + c) + c(a + b + c)\\ &= a^2 + ab + ac + ab + b^2 + bc + ac + bc + c^2\\ &= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc\end{aligned}$$

Underlying this expansion is the notion of a *partition* into categories. The example partitions two 'items' among three categories. In this case the categories are the variables a, b, and c and the items are exponents. For example the partition, {1, 0, 1} represents the term ac in the expansion, which could also be written as $a^1b^0c^1$, whereas the partition {0, 2, 0} represents the b^2 term. The coefficients of each term can be computed by the formula:

$\dfrac{n!}{n_1! \cdot n_2! \cdot n_3! \cdots n_k!}$, where n is the exponent and $n_1 + n_2 + n_3 + \cdots + n_k = n$. Recall, $0! = 1$.

Term	Partition	Coefficient
a^2	{2,0,0}	$\frac{2!}{2! \cdot 0! \cdot 0!} = 1$
b^2	{0,2,0}	$\frac{2!}{0! \cdot 2! \cdot 0!} = 1$
c^2	{0,0,2}	$\frac{2!}{0! \cdot 0! \cdot 2!} = 1$
ab	{1,1,0}	$\frac{2!}{1! \cdot 1! \cdot 0!} = 2$
ac	{1,0,1}	$\frac{2!}{1! \cdot 0! \cdot 1!} = 2$
bc	{0,1,1}	$\frac{2!}{0! \cdot 1! \cdot 1!} = 2$

1. Determine the all the *partitions* of $(x + y + z)^3$.

2. Determine the coefficients in the expansion of $(x + y + z)^3$ associated with each partition.

3. How can you build and interpret a trinomial distribution?

Lesson 12-9

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

12-10 Enrichment

Sample Mean and Standard Error

The mean of a sample of size n is an estimate of the mean of the entire population under study. A different sample of size n probably will have a different mean, and a third sample yet another. For example, suppose the means for three different samples were 9, 5, and 11. What prediction would you make for the mean of a fourth sample? Clearly there is some uncertainty involved in the prediction. Therefore the sample mean must have a probability distribution associated with it. If the standard deviation of the population is known, σ (sigma), then the standard deviation of the sample mean, called the *standard error*, is given by: $SE = \frac{\sigma}{\sqrt{n}}$, where n is the sample size.

A 95% confidence interval for the population mean is given by:
$(\overline{x} - 2SE, \overline{x} + 2SE) = \left(\overline{x} - 2\frac{\sigma}{\sqrt{n}}, \overline{x} + 2\frac{\sigma}{\sqrt{n}}\right)$. That is, there is a 95% chance that the *true* population mean will be contained in this interval.

Suppose a high school football team has 50 players. The coach wants to know what the average amount a player can bench press. The only exact way to calculate this average would be to measure, in the weight room, each player on the team. Instead the coach randomly samples 10 players and computes the average of these 10 to estimate the mean bench press weight for the entire team. Suppose the coach knew that $\sigma = 41.6$. (This data is unknown to the coach but presented here for the purpose of the exercises.)

185	185	200	205	215
235	240	200	205	180
175	190	195	215	205
250	275	250	230	250
205	260	190	210	230
180	175	165	165	195
290	300	305	315	210
315	210	230	205	200
150	155	145	200	215
160	190	195	185	180

Exercises

1. Calculate the standard error of the sample mean.

2. Calculate a 95% confidence interval for the population mean.

3. How can the coach make more accurate estimations?

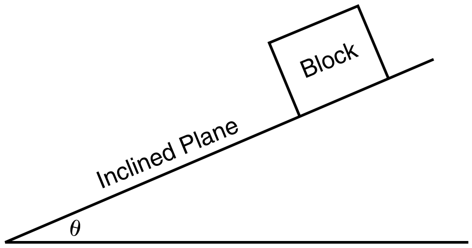
4. Calculate the sample mean and standard error for a sample of size 20.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

13-1 Enrichment

The Angle of Repose

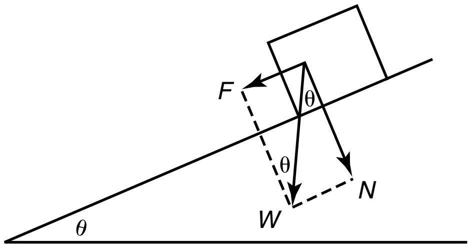
Suppose you place a block of wood on an inclined plane, as shown at the right. If the angle, θ, at which the plane is inclined from the horizontal is very small, the block will not move. If you increase the angle, the block will eventually overcome the force of friction and start to slide down the plane.



At the instant the block begins to slide, the angle formed by the plane is called the angle of friction, or the angle of repose.

For situations in which the block and plane are smooth but unlubricated, the angle of repose depends *only* on the types of materials in the block and the plane. The angle is independent of the area of contact between the two surfaces and of the weight of the block.

The drawing at the right shows how to use vectors to find a coefficient of friction. This coefficient varies with different materials and is denoted by the Greek letter mu, μ.



$$F = W \sin \theta \qquad N = W \cos$$
$$F = \mu N$$
$$\mu = \frac{\sin \theta}{\cos \theta} = \tan \theta$$

Solve each problem.

Material	Coefficient of Friction μ
Wood on wood	0.5
Wood on stone	0.5
Rubber tire on dry concrete	1.0
Rubber tire on wet concrete	0.7

1. A wooden chute is built so that wooden crates can slide down into the basement of a store. What angle should the chute make in order for the crates to slide down at a constant speed?

2. Will a 100-pound wooden crate slide down a stone ramp that makes an angle of 20° with the horizontal? Explain your answer.

3. If you increase the weight of the crate in Exercise 2 to 300 pounds, does it change your answer?

4. A car with rubber tires is being driven on dry concrete pavement. If the car tires spin without traction on a hill, how steep is the hill?

5. For Exercise 4, does it make a difference if it starts to rain? Explain your answer.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

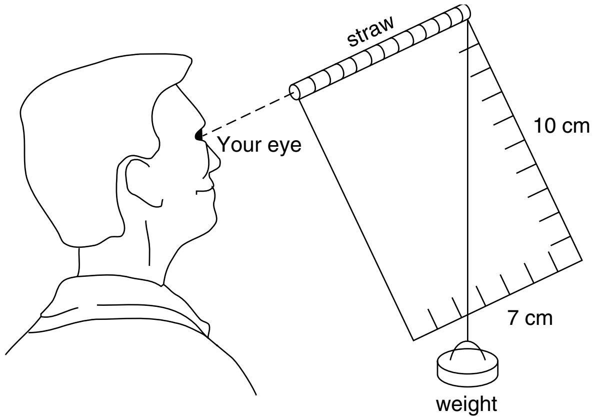
NAME ____ DATE______________ PERIOD _____

13-2 Enrichment

Making and Using a Hypsometer

A **hypsometer** is a device that can be used to measure the height of an object. To construct your own hypsometer, you will need a rectangular piece of heavy cardboard that is at least 7 cm by 10 cm, a straw, transparent tape, a string about 20 cm long, and a small weight that can be attached to the string.

Mark off 1-cm increments along one short side and one long side of the cardboard. Tape the straw to the other short side. Then attach the weight to one end of the string, and attach the other end of the string to one corner of the cardboard, as shown in the figure below. The diagram below shows how your hypsometer should look.



To use the hypsometer, you will need to measure the distance from the base of the object whose height you are finding to where you stand when you use the hypsometer.

Sight the top of the object through the straw. Note where the free-hanging string crosses the bottom scale. Then use similar triangles to find the height of the object.

1. Draw a diagram to illustrate how you can use similar triangles and the hypsometer to find the height of a tall object.

Use your hypsometer to find the height of each of the following.

2. your school's flagpole

3. a tree on your school's property

4. the highest point on the front wall of your school building

5. the goal posts on a football field

6. the hoop on a basketball court

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

13-3 Enrichment

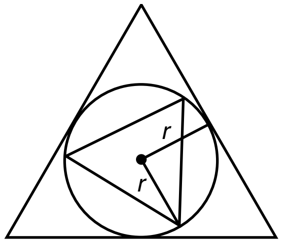
Areas of Polygons and Circles

A regular polygon has sides of equal length and angles of equal measure. A regular polygon can be inscribed in or circumscribed about a circle. For n-sided regular polygons, the following area formulas can be used.

Area of circle $A_C = \pi r^2$

Area of inscribed polygon $A_I = \frac{nr^2}{2} \times \sin \frac{360°}{n}$

Area of circumscribed polygon $A_C = nr^2 \times \tan \frac{180°}{n}$



Use a calculator to complete the chart below for a unit circle (a circle of radius 1).

	Number of Sides	Area of Inscribed Polygon	Area of Circle minus Area of Polygon	Area of Circumscribed Polygon	Area of Polygon minus Area of Circle
	3	1.2990381	1.8425545	5.1961524	2.054597
1.	4				
2.	8				
3.	12				
4.	20				
5.	24				
6.	28				
7.	32				
8.	1000				

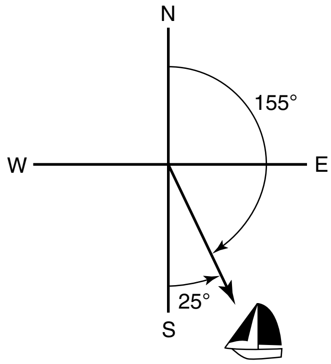
9. What number do the areas of the circumscribed and inscribed polygons seem to be approaching?

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

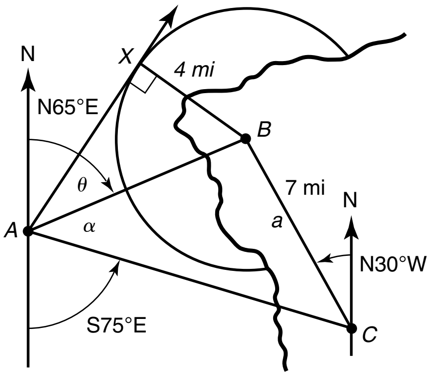
13-4 Enrichment

Navigation

The bearing of a boat is an angle showing the direction the boat is heading. Often, the angle is measured from north, but it can be measured from any of the four compass directions. At the right, the bearing of the boat is 155°. Or, it can be described as 25° east of south (S25°E).



Example **A boat A sights the lighthouse B in the direction N65°E and the spire of a church C in the direction S75°E. According to the map, B is 7 miles from C in the direction N30°W. In order for A to avoid running aground, find the bearing it should keep to pass B at 4 miles distance.**



In $\triangle ABC$, $\angle\alpha = 180° - 65° - 75°$ or $40°$

$\angle C = 180° - 30° - (180° - 75°)$

$= 45°$

$a = 7$ miles

With the Law of Sines,

$$AB = \frac{a \sin C}{\sin \alpha} = \frac{7(\sin 45°)}{\sin 40°} = 7.7 \text{ mi.}$$

The ray for the correct bearing for A must be tangent at X to circle B with radius $BX = 4$. Thus $\triangle ABX$ is a right triangle.

Then $\sin \theta = \frac{BX}{AB} = \frac{4}{7.7} \approx 0.519$. Therefore, $\angle\theta = 31°18'$.

The bearing of A should be $65° - 31°18'$ or $33°42'$.

Solve the following.

1. Suppose the lighthouse B in the example is sighted at S30°W by a ship P due north of the church C. Find the bearing P should keep to pass B at 4 miles distance.

2. In the fog, the lighthouse keeper determines by radar that a boat 18 miles away is heading to the shore. The direction of the boat from the lighthouse is S80°E. What bearing should the lighthouse keeper radio the boat to take to come ashore 4 miles south of the lighthouse?

3. To avoid a rocky area along a shoreline, a ship at M travels 7 km to R, bearing $22°15'$, then 8 km to P, bearing $68°30'$, then 6 km to Q, bearing $109°15'$. Find the distance from M to Q.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

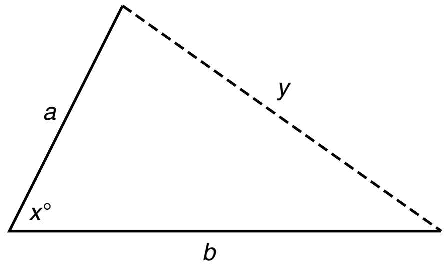
NAME ____ DATE______________ PERIOD _____

13-5 Enrichment

The Law of Cosines and the Pythagorean Theorem

The law of cosines bears strong similarities to the Pythagorean theorem. According to the law of cosines, if two sides of a triangle have lengths a and b and if the angle between them has a measure of $x°$, then the length, y, of the third side of the triangle can be found by using the equation

$y^2 = a^2 + b^2 - 2ab \cos x°$.



Answer the following questions to clarify the relationship between the law of cosines and the Pythagorean theorem.

1. If the value of $x°$ becomes less and less, what number is $\cos x°$ close to?

2. If the value of $x°$ is very close to zero but then increases, what happens to $\cos x°$ as $x°$ approaches 90°?

3. If $x°$ equals 90°, what is the value of $\cos x°$? What does the equation of $y^2 = a^2 + b^2 - 2ab \cos x°$ simplify to if $x°$ equals 90°?

4. What happens to the value of $\cos x°$ as $x°$ increases beyond 90° and approaches 180°?

5. Consider some particular value of a and b, say 7 for a and 19 for b. Use a graphing calculator to graph the equation you get by solving $y^2 = 7^2 + 19^2 - 2(7)(19) \cos x°$ for y.

a. In view of the geometry of the situation, what range of values should you use for X?

b. Display the graph and use the TRACE function. What do the maximum and minimum values appear to be for the function?

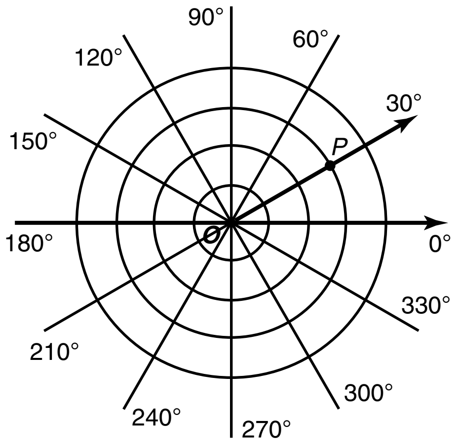
c. How do the answers for part **b** relate to the lengths 7 and 19? Are the maximum and minimum values from part **b** ever actually attained in the geometric situation?

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

13-6 Enrichment

Polar Coordinates

Consider an angle in standard position with its vertex at a point O called the *pole*. Its initial side is on a coordinated axis called the *polar axis*. A point P on the terminal side of the angle is named by the *polar coordinates* (r, θ) where r is the directed distance of the point from O and θ is the measure of the angle.



Graphs in this system may be drawn on polar coordinate paper such as the kind shown at the right.

The polar coordinates of a point are not unique. For example, $(3, 30°)$ names point P as well as $(3, 390°)$. Another name for P is $(-3, 210°)$. Can you see why? The coordinates of the pole are $(0, \theta)$ where θ may be any angle.

Example **Draw the graph of the function $r = \cos \theta$. Make a table of convenient values for θ and r. Then plot the points.**

θ	0°	30°	60°	90°	120°	150°	180°
r	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1

Since the period of the cosine function is 180°, values of r for $\theta > 180°$ are repeated.

Graph each function by making a table of values and plotting the values on polar coordinate paper.

1. $r = 4$

2. $r = 3 \sin \theta$

3. $r = 3 \cos 2\theta$

4. $r = 2(1 + \cos \theta)$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

13-7 Enrichment

Snell's Law

Snell's Law describes what happens to a ray of light that passes from air into water or some other substance. In the figure, the ray starts at the left and makes an angle of incidence θ with the surface.

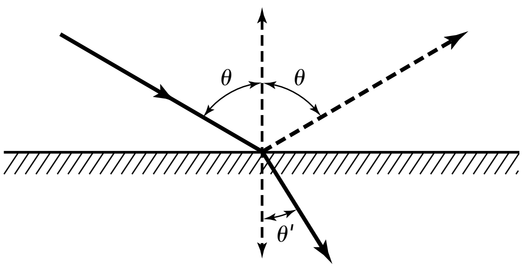
Part of the ray is reflected, creating an angle of reflection θ. The rest of the ray is bent, or refracted, as it passes through the other medium. This creates angle θ'.

The angle of incidence equals the angle of reflection.

The angles of incidence and refraction are related by Snell's Law:

$\sin \theta = k \sin \theta'$

The constant k is called the index of refraction.



k	Substance
1.33	Water
1.36	Ethyl alcohol
1.54	Rock salt and Quartz
1.46–1.96	Glass
2.42	Diamond

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

Use Snell's Law to solve the following. Round angle measures to the nearest tenth of a degree.

1. If the angle of incidence at which a ray of light strikes the surface of a window is 45° and $k = 1.6$, what is the measure of the angle of refraction?

2. If the angle of incidence of a ray of light that strikes the surface of water is 50°, what is the angle of refraction?

3. If the angle of refraction of a ray of light striking a quartz crystal is 24°, what is the angle of incidence?

4. The angles of incidence and refraction for rays of light were measured five times for a certain substance. The measurements (one of which was in error) are shown in the table. Was the substance glass, quartz, or diamond?

θ	15°	30°	40°	60°	80°
θ'	9.7°	16.1°	21.2°	28.6°	33.2°

5. If the angle of incidence at which a ray of light strikes the surface of ethyl alcohol is 60°, what is the angle of refraction?

14-1 Enrichment

Blueprints

Interpreting blueprints requires the ability to select and use trigonometric functions and geometric properties. The figure below represents a plan for an improvement to a roof. The metal fitting shown makes a 30° angle with the horizontal. The vertices of the geometric shapes are *not* labeled in these plans. Relevant information must be selected and the appropriate function used to find the unknown measures.

Example **Find the unknown measures in the figure at the right.**

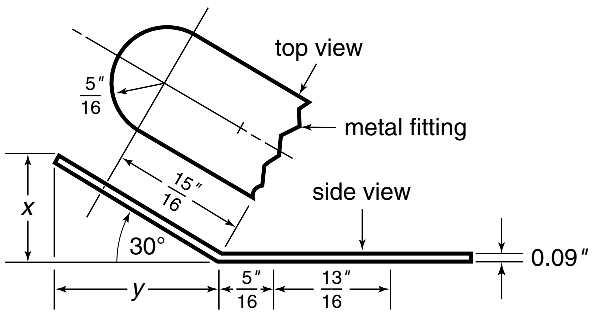
The measures x and y are the legs of a right triangle.

The measure of the hypotenuse is $\frac{15}{16}$ in. + $\frac{5}{16}$ in. or $\frac{20}{16}$ in.

$$\frac{y}{\frac{20}{16}} = \cos 30° \qquad \frac{x}{\frac{20}{16}} = \sin 30°$$

$$y = 1.08 \text{ in.} \qquad x = 0.63 \text{ in.}$$

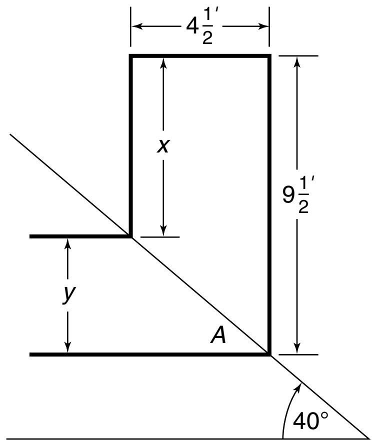
Roofing Improvement



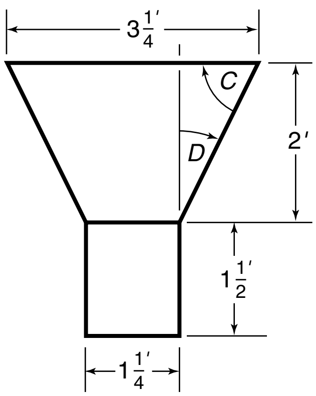
Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

Find the unknown measures of each of the following.

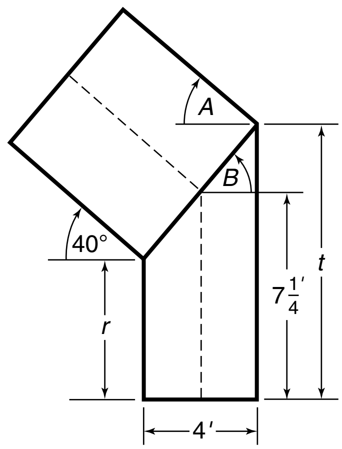
1. Chimney on roof



2. Air vent



3. Elbow joint



NAME ____ DATE______________ PERIOD _____

14-2 Enrichment

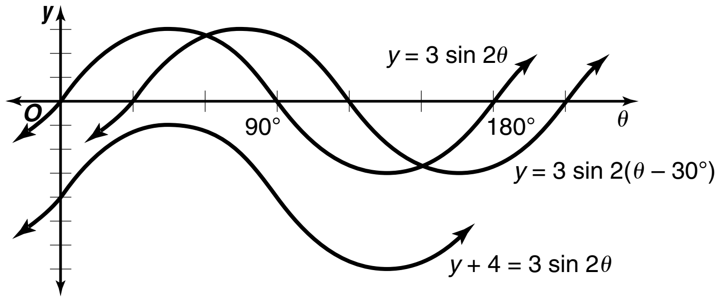
Translating Graphs of Trigonometric Functions

Three graphs are shown at the right:

$y = 3 \sin 2\theta$

$y = 3 \sin 2(\theta - 30°)$

$y + 4 = 3 \sin 2\theta$



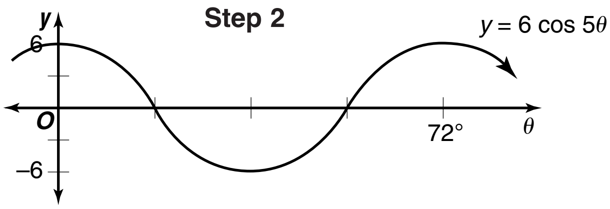
Replacing θ with $(\theta - 30°)$ translates the graph to the right. Replacing y with $y + 4$ translates the graph 4 units down.

Example **Graph one cycle of $y = 6 \cos (5\theta + 80°) + 2$.**

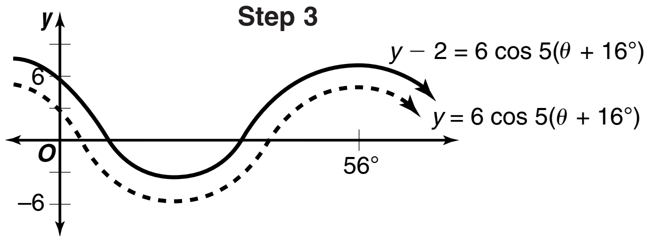
Step 1 Transform the equation into the form $y - k = a \cos b(\theta - h)$.

$y - 2 = 6 \cos 5(\theta + 16°)$

Step 2 Sketch $y = 6 \cos 5\theta$.



Step 3 Translate $y = 6 \cos 5\theta$ to obtain the desired graph.



Sketch these graphs on the same coordinate system.

1. $y = 3 \sin 2(\theta + 45°)$ **2.** $y - 1 = 3 \sin 2\theta$ **3.** $y + 5 = 3 \sin 2(\theta + 90°)$

On another piece of paper, graph one cycle of each curve.

4. $y = 2 \sin 4(\theta - 50°)$

5. $y = 5 \sin (3\theta + 90°)$

6. $y = 6 \cos (4\theta + 360°) + 3$

7. $y = 6 \cos 4\theta + 3$

8. The graphs for problems 6 and 7 should be the same. Use the sum formula for cosine of a sum to show that the equations are equivalent.

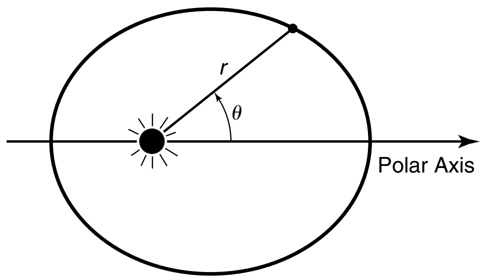
Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ____ DATE______________ PERIOD _____

14-3 Enrichment

Planetary Orbits

The orbit of a planet around the sun is an ellipse with the sun at one focus. Let the pole of a polar coordinate system be that focus and the polar axis be toward the other focus. The polar equation of an ellipse is



$r = \frac{2ep}{1 - e \cos \theta}$. Since $2p = \frac{b^2}{c}$ and $b^2 = a^2 - c^2$,

$2p = \frac{a^2 - c^2}{c} = \frac{a^2}{c}\left(1 - \frac{c^2}{a^2}\right)$. Because $e = \frac{c}{a}$,

$2p = a\left(\frac{a}{c}\right)\left(1 - \left(\frac{c}{a}\right)^2\right) = a\left(\frac{1}{e}\right)(1 - e^2)$.

Therefore, $2ep = a(1 - e^2)$. Substituting into the polar equation of an ellipse yields an equation that is useful for finding distances from the planet to the sun.

$$r = \frac{a(1 - e^2)}{1 - e \cos \theta}$$

Note that e is the eccentricity of the orbit and a is the length of the semi-major axis of the ellipse. Also, a is the mean distance of the planet from the sun.

Example **The mean distance of Venus from the sun is 67.24×10^6 miles and the eccentricity of its orbit is .006788. Find the minimum and maximum distances of Venus from the sun.**

The minimum distance occurs when $\theta = \pi$.

$$r = \frac{67.24 \times 10^6(1 - 0.006788^2)}{1 - 0.006788 \cos \pi} = 66.78 \times 10^6 \text{ miles}$$

The maximum distance occurs when $\theta = 0$.

$$r = \frac{67.24 \times 10^6(1 - 0.006788^2)}{1 - 0.006788 \cos 0} = 67.70 \times 10^6 \text{ miles}$$

Complete each of the following.

1. The mean distance of Mars from the sun is 141.64×10^6 miles and the eccentricity of its orbit is 0.093382. Find the minimum and maximum distances of Mars from the sun.

2. The minimum distance of Earth from the sun is 91.445×10^6 miles, and the eccentricity of its orbit is 0.016734. Find the mean and maximum distances of Earth from the sun.

Lesson 14-3

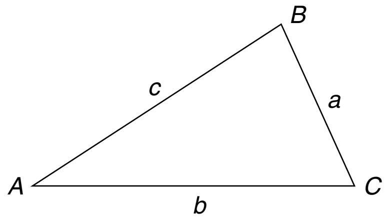
Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

NAME ______________________________ DATE ____________ PERIOD _____

14-4 Enrichment

Heron's Formula

Heron's formula can be used to find the area of a triangle if you know the lengths of the three sides. Consider any triangle ABC. Let K represent the area of $\triangle ABC$. Then



$$K = \frac{1}{2}bc \sin A$$

$$K^2 = \frac{b^2c^2 \sin^2 A}{4} \quad \text{Square both sides.}$$

$$= \frac{b^2c^2(1 - \cos^2 A)}{4}$$

$$= \frac{b^2c^2(1 + \cos A)(1 - \cos A)}{4}$$

$$= \frac{b^2c^2}{4}\left(1 + \frac{b^2 + c^2 - a^2}{2bc}\right)\left(1 - \frac{b^2 + c^2 - a^2}{2bc}\right) \quad \text{Use the law of cosines.}$$

$$= \frac{b + c + a}{2} \cdot \frac{b + c - a}{2} \cdot \frac{a + b - c}{2} \cdot \frac{a - b + c}{2} \quad \text{Simplify.}$$

Let $s = \frac{a + b + c}{2}$. Then $s - a = \frac{b + c - a}{2}$, $s - b = \frac{a + c - b}{2}$, $s - c = \frac{a + b - c}{2}$.

$$K^2 = s(s - a)(s - b)(s - c) \quad \text{Substitute.}$$

$$K = \sqrt{s(s - a)(s - b)(s - c)}$$

Heron's Formula	The area of $\triangle ABC$ is $\sqrt{s(s - a)(s - b)(s - c)}$, where $s = \frac{a + b + c}{2}$.

Use Heron's formula to find the area of $\triangle ABC$.

1. $a = 3, b = 4.4, c = 7$

2. $a = 8.2, b = 10.3, c = 9.5$

3. $a = 31.3, b = 92.0, c = 67.9$

4. $a = 0.54, b = 1.32, c = 0.78$

5. $a = 321, b = 178, c = 298$

6. $a = 0.05, b = 0.08, c = 0.04$

7. $a = 21.5, b = 33.0, c = 41.7$

8. $a = 2.08, b = 9.13, c = 8.99$

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

14-5 Enrichment

Identities for the Products of Sines and Cosines

By adding the identities for the sines of the sum and difference of the measures of two angles, a new identity is obtained.

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

(i) $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$

This new identity is useful for expressing certain products as sums.

Example **Write $\sin 3\theta \cos\theta$ as a sum.**

In the identity let $\alpha = 3\theta$ and $\beta = \theta$ so that
$2\sin 3\theta\cos\theta = \sin(3\theta + \theta) + \sin(3\theta - \theta)$. Thus,
$\sin 3\theta\cos\theta = \frac{1}{2}\sin 4\theta + \frac{1}{2}\sin 2\theta$.

By subtracting the identities for $\sin(\alpha + \beta)$ and $\sin(\alpha - \beta)$, a similar identity for expressing a product as a difference is obtained.

(ii) $\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos\alpha\sin\beta$

Solve.

1. Use the identities for $\cos(\alpha + \beta)$ and $\cos(\alpha - \beta)$ to find identities for expressing the products $2\cos\alpha\cos\beta$ and $2\sin\alpha\sin\beta$ as a sum or difference.

2. Find the value of $\sin 105° \cos 75°$ without using tables.

3. Express $\cos\theta \sin\frac{\theta}{2}$ as a difference.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

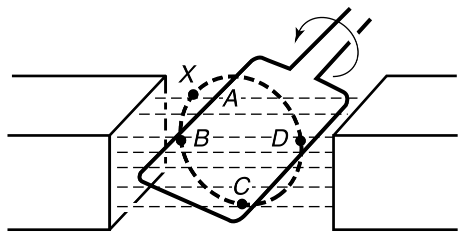
Lesson 14-5

NAME ______________________________ DATE______________ PERIOD _____

14-6 Enrichment

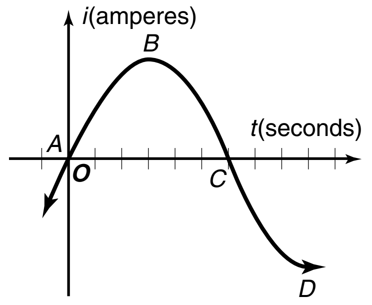
Alternating Current

The figure at the right represents an alternating current generator. A rectangular coil of wire is suspended between the poles of a magnet. As the coil of wire is rotated, it passes through the magnetic field and generates current.



As point X on the coil passes through the points A and C, its motion is along the direction of the magnetic field between the poles. Therefore, no current is generated. However, through points $B$and D, the motion of X is perpendicular to the magnetic field. This induces maximum current in the coil. Between A and B, B and C, C and D, and D and A, the current in the coil will have an intermediate value. Thus, the graph of the current of an alternating current generator is closely related to the sine curve.

The maximum current may have a positive or negative value.



The actual current, i, in a household current is given by $i = I_M \sin(120\pi t + \alpha)$ where I_M is the maximum value of the current, t is the elapsed time in seconds, and α is the angle determined by the position of the coil at time t_n.

Example **If $\alpha = \frac{\pi}{2}$, find a value of t for which $i = 0$.**

If $i = 0$, then $I_M \sin(120\pi t + \alpha) = 0$. $\quad$ $i = I_M \sin(120\pi t + \alpha)$

Since $I_M \neq 0$, $\sin(120\pi t + \alpha) = 0$. $\quad$ If $ab = 0$ and $a \neq 0$, then $b = 0$.

Let $120\pi t + \alpha = s$. Thus, $\sin s = 0$.

$s = \pi$ because $\sin \pi = 0$.

$120\pi t + \alpha = \pi$ $\quad$ Substitute $120\pi t + \alpha$ for s.

$120\pi t + \frac{\pi}{2} = \pi$ $\quad$ Substitute $\frac{\pi}{2}$ for α.

$= \frac{1}{240}$ $\quad$ Solve for t.

This solution is the first positive value of t that satisfies the problem.

Using the equation for the actual current in a household circuit, $i = I_M \sin(120\pi t + \alpha)$, solve each problem. For each problem, find the first positive value of t.

1. If $\alpha = 0$, find a value of t for which $i = 0$

2. If $\alpha = 0$, find a value of t for which $i = +I_M$.

3. If $\alpha = \frac{\pi}{2}$, find a value of t for which $i = -I_M$.

4. If $\alpha = \frac{\pi}{4}$, find a value of t for which $i = 0$.

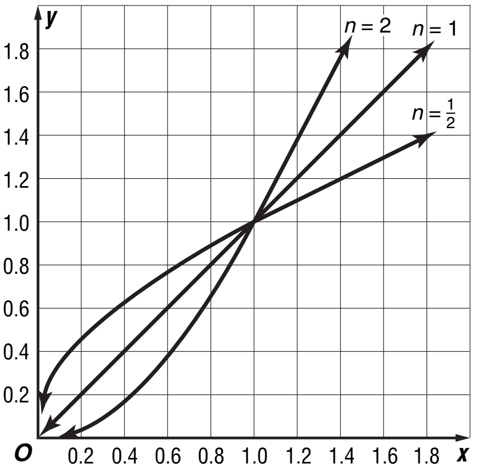
Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.

14-7 Enrichment

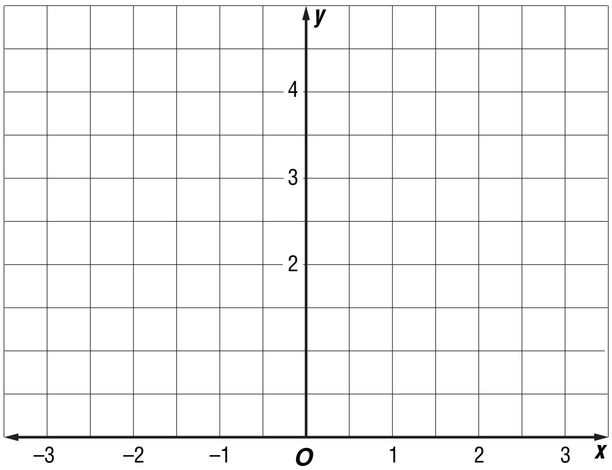
Families of Curves

Use these graphs for the problems below.

The Family $y = x^n$



The Family $y = e^{mx}$



1. Use the graph on the left to describe the relationship among the curves $y = x^{\frac{1}{2}}$, $y = x^1$, and $y = x^2$.

2. Graph $y = x^n$ for $n = \frac{1}{10}, \frac{1}{4}$, 4, and 10 on the grid with $y = x^{\frac{1}{2}}$, $y = x^1$, and $y = x^2$.

3. Which two regions in the first quadrant contain no points of the graphs of the family for $y = x^n$?

4. On the right grid, graph the members of the family $y = e^{mx}$ for which $m = 1$ and $m = -1$.

5. Describe the relationship among these two curves and the y-axis.

6. Graph $y = e^{mx}$ for $m = 0, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm2$, and ±4.

Copyright © Glencoe/McGraw-Hill, a division of The McGraw-Hill Companies, Inc.